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Cassirer's Reception of Dedekind and the Structuralist Transformation of Mathematics

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For much of the 20th century, Ernst Cassirer was seen as an intellectual historian, besides being the last member of Marburg Neo-Kantianism. More recently, he has been rediscovered as an original, substantive philosopher in his own right, perhaps even one of the great philosophers of the 20th century. This concerns his contributions to the philosophy of science (relativity theory, quantum mechanics, etc.), his mature, wide-ranging philosophy of symbolic forms (leading to a "philosophy of culture"), and the ways in which his views position him, in potentially fruitful ways, at the intersection of "analytic" and "continental" philosophy.[1] In addition, Cassirer was a keen observer of developments in pure mathematics, especially of their philosophical significance. There are two separable, though not unrelated, strands on that topic in his writings. The first concerns his reflections on revolutionary changes in geometry during the 19th century, culminating in David Hilbert's and Felix Klein's works. The second strand involves parallel changes in algebra, arithmetic, and set theory, where Evariste Galois, Richard Dedekind, and Georg Cantor played key roles.

In this essay the main focus will be on the second of the strands just mentioned, and in particular, on Cassirer's reception of Dedekind, which still deserves more attention.[2] As we will see, Cassirer was a perceptive reader of Dedekind, arguably still his subtlest philosophical interpreter. What he was concerned about with

[1] For the philosophy of physics, cf. Ryckman (2005) and French (2014), more generally also Ihmig (2001) and Part I of Friedman and Luft (2017); for the philosophy of culture, cf. Recki (2004), Luft (2015), and Part III of Friedman and Luft (2017); for Cassirer's position between the analytic and continental traditions, cf. Friedman (2000) and Part II of Friedman and Luft (2017). For more general discussions, see also Ferrari (2003) and Kreis (2010).

[2] Cf. Reck (2013), chapter 4 of Biagioli (2016), Yap (2017), and Heis (2017) for some recent discussions of the topic. For Cassirer's views on geometry, Klein's Erlangen program, and Hilbert's axiomatics, which have found more attention in the literature already, cf. Ihmig (1997, 1999), Mormann (2007), Heis (2011), most of Biagioli (2016), and Schiemer (2018). I will come back to the latter briefly later in this essay.

respect to mathematics in general was the introduction of "ideal elements", together with related, very significant expansions of its scope over time. This led to a reconsideration of its subject matter, including the rejection of the traditional view that mathematics is "the science of quantity and number". Cassirer's discussion of this topic often took place under the label of "concept formation" in science; and he identified a corresponding shift from "substance concepts" to "function concepts", seen as culminating in the 19th and early 20th centuries. What he arrived at with such considerations was, in later terminology, a structuralist conception of mathematical objects; and that conception was rooted in observations about mathematical methodology. Dedekind's work was decisive for Cassirer since he saw in it the clearest and most powerful example of the structuralist "unfolding" of mathematics, i.e., of the systematic, mature development of older structuralist "germs" in it.[3]

This essaywill proceed as follows: first, an outline of Cassirer's overall perspective on mathematics will be provided. In the second section, we will turn to a brief summary of Dedekind's relevant contributions, one in which their crucial but also controversial structuralist dimension will be highlighted. Third, we will see how Cassirer's sympathetic reception of Dedekind's structuralism contrasts sharply with criticisms and dismissals by other philosophers, starting with Frege and Russell. This will lead to some historically grounded and philosophically significant observations about "existence," "determinateness," and "givenness" in modern mathematics. In the fourth section, several aspects of Cassirer's own views about structuralism, related to but also going beyond Dedekind, will be discussed. The latter will include a deeper motivation for structuralism than is usually provided today; some original views about the role of constructions in structuralist mathematics, together with its historical "unfolding"; and his insistence on the fact that the metaphysics and the methodology of mathematics, or of any science for that matter, should be viewed as inseparable. A brief conclusion will round off the essay.

1. Cassirer's Overall Perspective on Mathematics

Dedekind's contributions to mathematics play a prominent role in Cassirer's writings from early on. The first clear expressions of this fact occurs in his survey article "Kant und die moderne Mathematik" (1907), the second in his first systematical book, *Substanzbegriff und Funktionsbegriff* (1910). Dedekind remains an important reference point later on, e.g., in *Die Philosophie der*

[3] A second aspect of Dedekind's work important for Cassirer was his logicism. While not unrelated, I will leave it largely aside here; cf. Reck and Keller (forthcoming) for more.

Symbolischen Formen, vol. 3 (1929) and in *The Problem of Knowledge*, vol. 4 (1950).[4] The general context is Cassirer's discussion, in the relevant parts of these works, of the rise of modern mathematics and mathematical science—from Kepler's, Galilei's, and Descartes's innovative "mathematization" of nature, through the introduction of the integral and differential calculus by Leibniz, Newton, and their followers, to a number of developments in the 19th century.

With respect to the emergence of modern mathematics, there are two main strands one can distinguish: the gradual acceptance and systematization of various new geometries (projective, elliptic and hyperbolic, etc.), leading to David Hilbert's formal axiomatics and Felix Klein's "Erlangen program"; and the parallel expansion and diversification of algebra and arithmetic (complex numbers, Galois theory, Hamilton's quaternions, new conceptions of the real numbers, etc.), which brought with it the rise of set theory (including Cantor's transfinite numbers) and the replacement of Aristotelian logic by modern mathematical logic (Boole, Frege, Peano, Russell, and others). One noteworthy component of both strands is the introduction and systematic use of "ideal elements" in modern mathematics, such as points at infinity in projective geometry or, earlier, the complex numbers.

Cassirer was not the only philosopher surveying and analyzing these developments at the time. In fact, in this respect he followed his teachers in the Marburg School: Hermann Cohen and Paul Natorp (cf. Cohen 1883 and Natorp 1910). However, both Cohen and Natorp make the concept of the infinitesimal central to their accounts of science, while Cassirer shifts to a different perspective. He fully accepts the "arithmetization of analysis" by Cauchy, Bolzano, Weierstrass, Cantor, Dedekind, and others, with its replacement of infinitesimals by the familiar ε-δ treatment of limits. Unlike Cohen and Natorp, he also emphasizes that set theory and modern logic are natural next steps in this development, just as Hilbert's and Klein's approaches are with respect to unifying the new geometries. Moreover, Cassirer explicitly endorses Cantor's and Dedekind's emphasis on "mathematical freedom", i.e., the fact that modern mathematics has gone far beyond what is suggested in applications to nature and is exploring radically new "conceptual possibilities".[5]

What all these developments require, if we want to account for them systematically, is a novel conception of mathematics with respect to both its methodology and its subject matter. In Cassirer's own words:

[4] Cassirer mentions Dedekind in other writings too, including his early book on Leibniz (1902), his monumental series, *Das Erkenntnisproblem*, vols. 1–3 (1906–1910), and some of his later works, e.g., *An Essay on Man* (1944). But Cassirer (1907, [1910] 1923, [1929] 1965, 1950) will be the main sources of evidence for me, since they contain the most relevant and extensive discussions.

[5] For the idea of "mathematical freedom," cf. Tait (1996), for the exploration of new "conceptual possibilities," Stein (1988). (While in line with Cassirer's approach, neither of them mentions him.)

> Mathematics is no longer—as it was thought of for centuries—the science of quantity and number, but henceforth encompasses all contents for which complete law-like determinateness and continuous deductive interconnection is achievable. (Cassirer 1907, 40, my trans.)

It should be clear what is given up here, namely the view of mathematics as "the science of quantity and number", with its roots going back to Euclid. But what does Cassirer have in mind when he writes about "complete law-like determinateness" and "deductive interconnection"? Presumably these are meant to encompass the new developments in geometry, algebra, and arithmetic already mentioned. But how exactly; and what are some specific examples?

As we will see soon, it is Dedekind's treatment of the natural numbers and the real numbers that serves as the new paradigm for Cassirer here. It is primarily, although not exclusively, with those examples in mind that he writes:

> Here we encounter for the first time a general procedure that is of decisive significance for the whole formation of mathematical concepts. Wherever a *system of conditions* is given that can be realized in different contents [*das sich in verschiedenen Inhalten erfüllen kann*], we can hold on to the form of the system as an *invariant*, putting aside the difference in contents, and develop its laws deductively. (Cassirer [1910] 1923, 40, trans. modified)

As a relevant "system of conditions", consider Dedekind's characterization of the real numbers in terms of the concept of a continuous ordered field; and as two ways of "realizing" these conditions, take Dedekind's construction via the system of cuts on the rational numbers and Cantor's alternative construction via (equivalent classes of) Cauchy sequences (more on both later). The "invariant" to which we hold on in this case is "the real numbers"; and we "develop their laws deductively" based on Dedekind's definitions. This, then, is a paradigm of "law-like determinateness" and "logical interconnection".

Cassirer does not call the resulting "invariant", or the system of abstract objects thereby characterized, a "structure". But he comes close, e.g., when he writes:

> In this way we produce a new "objective" formation [*Gebilde*] whose structure [*Struktur*] is independent of all arbitrariness. But it would be uncritical naïveté to confuse the object thus arising with sensuously real and actual things. We

> cannot read off its "properties" empirically; nor do we need to, for it is revealed in all its determinateness as soon as we have grasped the relation from which it develops in all its purity. (1910, 40–41, trans. modified)

In the example just used, the "objective formation" is the system of real numbers as introduced by Dedekind—which "has", or alternatively "is", a certain structure. The fact that its "determinateness" is independent of empirical facts corresponds to the "mathematical freedom" Dedekind and Cantor emphasized. And the resulting "purity" has to do with the fact that all of this can be done in "pure logic" for both Dedekind and Cassirer. Finally, what is crucial about this conception of mathematics for Cassirer is that it is applicable equally to older, seemingly concrete parts of mathematics, such as elementary arithmetic or Euclidean geometry, and to novel, more abstract parts involving "ideal elements", e.g., complex numbers and points at infinity—both can now be understood as concerning (relational or functional) structures. Along such lines, pure mathematics in its entirety concerns "ideal" objects.

In Cassirer's 1910 book, *Substanzbegriff und Funktionsbegriff*, the conception of pure mathematics and mathematical science that results is characterized as involving "function concepts", as opposed to "substance concepts". Before examining further how he understands that distinction, let me complete my initial survey of Cassirer's perspective on mathematics throughout his career. While the focus in his 1910 book is on "scientific cognition", Cassirer broadens his point of view considerably during the 1920s and 1930s, by developing his wide-ranging philosophy of symbolic forms. Basically, a "symbolic form" is a way of "objectifying" various things, or better, a way of "constituting" both subjects and objects; and Cassirer now identifies several of them as integral parts of human culture.[6] The symbolic form at play in mathematical science, especially in its modern shape, remains a prime example (in some sense the most advanced example, although all are interdependent in the end); but there is also a variety of others, including mythical and religious thought, ordinary language and ordinary knowledge, art, history, law, technology, etc. (in an open-ended list).

According to Cassirer's mature position, human thought always involves symbolic processes, i.e., various ways of determining, constituting, and presenting things, be it in science or in other cultural spheres. The primary foil in this connection, i.e., the view to which he is fundamentally opposed, is a kind of naive realism according to which objects are simply "given" to subjects in experience, without any symbolic mediation or constitution (with nature already "cut at its

[6] What exactly a "symbolic form" is, or how Cassirer understands the underlying notion of "symbol," is a complex question. Roughly, a "symbolic form" is a system of signs, rules, and practices used to represent, and constitute in the first place, aspects of the world or of oneself. For more, cf. Cassirer (1923, 1927, [1929] 1965), Ferrari (2003), chapter 6, and much of Kreis (2010).

joints", as it were). Cassirer follows in Kant's footsteps in this respect, and more specifically, his "critical" approach to philosophy" (in the form adopted by the Marburg School).[7] According to how he develops that position further, from the 1920s on, his focus on the symbolic constitution of subjects and objects requires close attention to logical and methodological issues.

Cassirer calls the general perspective that results "logical idealism". With his original example of mathematics in the foreground (although the core points apply more generally), he characterizes it as follows:

> Logical idealism starts from an analysis of mathematical "objects" and seeks to apprehend the peculiar determinacy of these objects by explaining them through the peculiarity of the mathematical "method," mathematical concept formation, and the formulation of its problems. (Cassirer [1929] 1965, 405, trans. modified slightly)

Cassirer's paradigm in the case of pure mathematics, i.e., his main inspiration and illustration, remains Dedekind (besides Cassirer [1929] 1965, cf. Cassirer 1950 and 1999). And it is to Dedekind's (methodological and metaphysical) structuralism that we now turn in more detail.

2. Dedekind's Structuralism and Its Critical Reception

The two texts by Dedekind on which Cassirer focuses, like most later philosophers of mathematics, are his 1872 essay, *Stetigkeit und irrationale Zahlen*, on the real numbers $\mathbb{R}$, and his 1888 essay, *Was sind und was sollen die Zahlen?*, on the natural numbers $\mathbb{N}$.[8] In both, Dedekind does exactly what we saw Cassirer highlight: he formulates "systems of conditions" that can be "realized in different contents"; and he considers a corresponding "objective formation", i.e., an abstract structure that is logically and fully determined by the system of conditions.

In the 1872 essay, the relevant "system of conditions"—which defines a (higher-order) concept—is that for a "continuous ordered field". Actually, Dedekind introduced the concept of a field (*Körper*) already earlier, in his writings on algebra and algebraic number theory.[9] What he adds now is the concept of continuity (*Stetigkeit*) (or line-completeness). Famously, the latter is defined in terms

[7] Kant's "Copernican Revolution" plays a central role here; cf. Keller (2015). With respect to my general understanding of Cassirer, and this point especially, I owe a big debt to Pierre Keller.

[8] Somewhat surprisingly, Cassirer does not comment on Dedekind's important contributions to algebra, algebraic number theory, etc., while he mentions some closely related works, e.g., by Galois and Hamilton. Cf. Reck (2016) for connections between all of Dedekind's contributions.

[9] Cf. the essay on Dedekind in the present volume, co-authroed by José Ferreirós and me.

of Dedekind's concept of cut. Dedekind then considers the system of all cuts on the rational numbers ℚ, endowed with a corresponding ordering and field operations (induced by those on ℚ), and he shows that that system is a continuous ordered field. He is well aware that alternative such systems can be constructed too, most prominently that of all (equivalence classes of) Cauchy sequences on ℚ, as Cantor and others had done. In other words, the "conditions" for being a continuous ordered field are "realized" by several systems. In a final step, Dedekind introduces "the real numbers" as a separate "pure" system corresponding to the system of cuts (isomorphic to but not identical with it); and he calls its introduction an act of "creation".[10]

Implicit in the procedure of Dedekind's 1872 essay, in the introduction of the system of cuts on ℚ, are two assumptions: first, that we have the (infinite) system of all rational numbers available; second, that we can perform certain "logical" or set-theoretic constructions on it (essentially by forming the power-set of ℚ). A main aim of Dedekind's 1888 essay, *Was sind und was sollen die Zahlen?*, is to provide a framework in which both of these assumptions can be justified further, i.e. a general theory of sets (*Systeme*) and functions (*Abbildungen*).[11] Within that framework, he then formulates another crucial "system of conditions", by defining the (higher-order) concept of a "simply infinite system". The latter depends, in turn, on several previously introduced concepts that are all "logical" (the concept of "infinity" and the more technical concept of "chain"). After that, he gives an argument that there are simply infinite systems (involving "thoughts", "thoughts of thoughts", etc.), parallel to his construction of the system of cuts on ℚ in 1872. And at that point, Dedekind adds a step not present in his earlier essay yet (although it can be supplemented retrospectively). Namely, he proves that any two simply infinite systems are isomorphic (his famous categoricity theorem). Finally, he uses both results to justify the "free creation"—via a process of "abstraction"—of a system that deserves to be called "the natural numbers".[12]

[10] In Dedekind's own words: "Whenever, then, we have to do with a cut (A_1, A_2) produced by no rational number, we create a new, an irrational number α, which we regard as completely defined by this cut (A_1, A_2); we shall say that the number α corresponds to this cut, or that it produces this cut" (Dedekind 1872, 15).

[11] The justification of the two assumptions mentioned remains implicit, however. Dedekind does not formulate basic laws for his set- and function-theoretic constructions; nor does he explicitly construct ℚ from ℕ, although he was familiar with how to do so. Cf. Reck (2003, 2016) for details.

[12] In Dedekind's own words again: "If in the consideration of a simply infinite system *N* set in order by a mapping *φ*, we entirely disregard the particular character of the elements, retaining merely their distinctness, and taking into account only the relations to one another in which they are placed by the order-setting mapping *φ*, then are these elements called *natural numbers* or *ordinal numbers* or simply *numbers*, and the base-element 1 is called the *base-number* of the *number-series N*. With reference to this freeing the elements from every other content (abstraction) we are justified in calling numbers a free creation of the human mind" (Dedekind 1888, 68).

The natural way to understand Dedekind's talk of "free creation" (although not an uncontroversial one) is the following: by a kind of "abstraction" we move from a previously constructed, relatively concrete system of objects (a particular continuous ordered field, a particular simply ordered system) to a new system that, while isomorphic, is distinct and more basic ("pure", more abstract, defined structurally). Understood as such, Dedekind's position amounts to a version of "non-eliminative structuralism" (in terminology introduced by Charles Parsons).[13] In the next section, I will provide further evidence that this is how Cassirer understands Dedekind, also that it is the position he adopts himself. It is striking, then, that most interpreters have had a very different, more critical reaction. Or rather, while almost all readers of Dedekind have accepted his technical contributions to the foundations of mathematics (his definitions of cut, continuity, infinity, simple infinity, his construction of the system of cuts, his categoricity theorem for simple infinities, etc.), his informal, more philosophical views about "abstraction" and "free creation", together with the resulting structuralism, have often been seen as problematic.

Bertrand Russell's critical reaction to Dedekind's structuralism is a good early illustration, one that was also highly influential. Basically, Russell could not make sense of objects introduced purely "relationally", like Dedekind's natural numbers, i.e., what Russell calls the finite "ordinals". As he puts it in his 1903 book, *The Principles of Mathematics*:

> IIt is impossible that the ordinals should be, as Dedekind suggests, nothing but the terms of such relations as constitute a progression. If they are to be anything at all, they must be intrinsically something. (Russell [1903] 1992, 249)

Russell assumes here, without further argument, that any object must have an "inner nature", one that goes beyond purely relational or structural properties (an assumption other philosophers have found plausible too). Hence, he finds the notion of an abstract or pure structure unintelligible. The other side of the coin is that he finds Dedekind's notion of "abstraction" unclear and unacceptable. In an attempt to be charitable, he concludes: "What Dedekind presents to us is not the numbers, but any progression [i.e., simply infinite system]: what he says is true of all progressions alike" (249). He then suggests using his own "principle of abstraction" instead, which amounts to the construction of the natural numbers in terms of equivalence classes of classes, as is well known. But this will

[13] Cf. Reck (2003) for further details. -

not do for Dedekind's purposes, because Russell's form of "abstraction" does not lead to a system isomorphic to the original one.[14]

A second philosopher whose early criticisms of Dedekind were quite influential is Gottlob Frege. In Frege's *Grundgesetze der Arithmetik*, volume 2, also published in 1903, he considers Dedekind's theory of the real numbers. He thereby lumps Dedekind with several other thinkers (Stolz, Heine, Cantor, etc.) who talk about the mental "creation" of mathematical objects. In Frege's view, this is problematic for at least two reasons: First, it seems to lead to a subjectivist, perhaps even solipsistic position in the end. Second, it is in danger of being inconsistent; and this is especially so if the "creation" at issue is not backed up by explicit principles or basic laws (like the ones Frege formulates for his own approach). Frege is also critical of Dedekind's talk of set formation in his 1888 essay in terms of "mental" operations, since he sees that as problematically psychologistic too. Finally, Frege and Russell take the application of the natural numbers as cardinal numbers to be more basic than their ordinal use. Their definition as cardinal numbers, in the form proposed by both of them, thus appears more justifiable and appropriate.[15]

Frege's and Russell's criticisms of Dedekind's views, especially of his remarks about "abstraction" and "free creation", produced many echoes in later philosophy. A particularly explicit and stark example occurs in Michael Dummett's 1995 book, *Frege: Philosophy of Mathematics*. In that book, both Frege and Russell are appealed to as authorities, specifically with their arguments just mentioned, in support of Dummett's claim that Dedekind's position amounts to "mystical structuralism"—clearly a position not to be taken seriously. Finally, even after the re-emergence of a variety of structuralist positions in the philosophy of mathematics from the 1960s on, the corresponding authors (Paul Benacerraf, Michael Resnik, Stewart Shapiro, Geoffrey Hellman, and others) have remained suspicious of Dedekind's original, seemingly psychologistic ways of putting things, while appropriating him as a predecessor more generally. In other words, even in the writings of self-proclaimed structuralists, Frege's and Russell's early criticisms still reverberate strongly.[16] This is in striking contrast to Cassirer's sympathetic reception of Dedekind, which we consider next.

[14] While Russell is dismissive of Dedekind's structuralist position in his 1903 book, unpublished manuscripts show that he was more sympathetic originally and that his dismissive stance was the last of several stages through which he went; cf. the essay by Heis in this collection for details.

[15] For more details concerning Frege's reaction to Dedekind, cf. Reck (2019). As suggested in that article, it might be possible to defend Dedekind by formulating both "construction" and "abstraction" principles for him, although it is questionable if this can be done "purely logically."

[16] Cf. Reck (2013) for further details on Dedekind's reception, including by Frege, Russell, Dummett, and later structuralists. In mathematics, his writings were received more positively, e.g., by Ernst Schröder, David Hilbert, Ernst Zermelo, and Emmy Noether. But this was also not universal; and his remarks about "abstraction" and "free creation" were often simply ignored in that context.

3. Cassirer's Sympathetic Reception of Dedekind

As previously mentioned, Cassirer takes Dedekind's approach to the natural and real numbers to be essentially correct, even paradigmatic, already in 1907, only a few years after Frege's and Russell's criticisms. He also defends Dedekind explicitly against their criticisms, arguing that his approach is superior to theirs. Concerning the natural numbers, this defense includes taking an "ordinal" approach to be as basic as, and in some respects more fundamental than, a "cardinal" approach. This leads Cassirer to the following remark:

> [Dedekind showed that] in order to provide a foundation for the whole of arithmetic, it is sufficient to define the number series simply as the succession of elements related to each other by means of a certain order—thereby thinking of the individual finite numbers, not as "pluralities of units," but as characterized merely by the "position" they occupy within the whole series. (Cassirer 1907, 46)

The conception of natural numbers as "pluralities of units" is the traditional one traceable back to Euclid. It constitutes both a "cardinal" approach and a "substance-based" view, in Cassirer's terminology. As such, it is inferior to, and to be replaced by, Dedekind's "ordinal" and "function-based" conception, in which the natural numbers are treated simply as "positions" in a series.

With this characterization of natural numbers as "positions", we have arrived at Cassirer's own structuralism. In his 1910 book, he adds the following about it:

> It becomes evident that the system of numbers as pure ordinal numbers can be derived immediately and without circuitous route through the concept of class; since for this we need to assume nothing but the possibility of differentiating a sequence of pure thought constructions by different relations to a determinate base element, which serves as a starting point. The theory of the ordinal numbers thus represents the essential minimum that no logical deduction of the concept of number can avoid. (Cassirer [1910] 1923, 53, trans. modified)

It is, of course, Frege and Russell who define the natural numbers "through the concept of class". One reason for seeing Dedekind's ordinal conception as superior is that, instead of using such a "circuitous route", it brings out "the essential minimum" on which arithmetic relies. (Cassirer's point is confirmed by the possibility, and now standard practice, of developing arithmetic simply based on the Dedekind-Peano axioms.) With respect to the real numbers, he remarks along related lines:

> We thus see that, to get to the concept of irrational number, we do not need to consider the intuitive geometric relationships of magnitudes, but can reach this goal entirely within the arithmetic realm. A number, considered purely as part of a certain ordered system, consists of nothing more than a "position." ([1910] 1923, 49, my trans.)

In this case the traditional conception, thus Cassirer's foil, starts from an appeal to intuitively given geometric magnitudes, a conception widely shared well into the 19th century. That conception gets replaced by Dedekind's purely "arithmetic", or even "logical", approach in terms of cuts, continuity, etc.

What Dedekind has thus provided, as noted by Cassirer explicitly, is "the essential conceptual characterization" for both $\mathbb{N}$ and $\mathbb{R}$ (1907, 53); and in doing so, he has provided the "logical foundations of the pure concept of number" ([1910] 1923, 35). This goes significantly beyond Frege's and Russell's class-based constructions, in his opinion. Remember also Cassirer's use of the term "position" (in the two preceding passages, among others) to describe the resulting conception. This is more than 50 years before, in the 1960s, Paul Benacerraf reopened the debate about structuralism in English-speaking philosophy of mathematics; and it is more than 70 years before, from the 1980s on, Michael Resnik, Stewart Shapiro, and others started to use that term prominently to characterize non-eliminative structuralism.[17]

Earlier we encountered Russell's core objection to a non-eliminative structuralist conception, based on his assumption that numbers, like all objects, must be "intrinsically something". Cassirer takes up this point directly, as follows:

> If the ordinal numbers are to be anything, they must—so it seems—have an "inner" nature and character; they must be distinguished from other entities by some absolute "mark," in the same way that points are different from instants, or tones from colors. But this objection mistakes the real aim and tendency of Dedekind's formation of concepts. What is at issue is just this: that there is a system of ideal objects whose content is exhausted in their mutual relations. The "essence" of the numbers consists in nothing more than their positional value. (Cassirer [1910] 1923, 39, trans. modified)

In Cassirer's eyes, Russell's view that objects have to be distinguished by some "absolute mark" is unwarranted. But more than that, it shows Russell to hold on

[17] Cf. Reck and Price (2000) for references. One may wonder whether there was a direct influence in this connection. I am not aware of any references to Cassirer in published works by Benacerraf, Resnik, or Shapiro. But in Benacerraf's dissertation (on logicism), Cassirer's use of "position" in his 1910 book is quoted in a footnote (Benacerraf 1960, 162), as pointed out to me by Sean Walsh.

to an older, obsolete, "substance-based" view (despite his commendable introduction of a "logic of relations", which Cassirer praises and adopts himself). This is what makes Russell's position, and similarly Frege's, less adequate to modern mathematics than Dedekind's.[18]

Cassirer responds to the psychologism change against Dedekind—raised by Frege and, more vehemently, by neo-Fregeans like Dummett—as well. As Cassirer understands him, Dedekind's appeal to "abstraction" and "free creation" should not be interpreted along problematic psychologistic lines. In fact:

> [In Dedekind's works] abstraction has, the effect of a liberation; it means logical concentration on the relational system, while rejecting all psychological accompaniments that may force themselves into the subjective stream of consciousness, which form no constitutive moment [*sachlich-konstitutives Moment*] of this system. ([1910] 1923, 39, trans. modified)

By taking Dedekind abstraction to involve "logical concentration on the relational system", Cassirer points to its logical and structural nature. This is what the critics, with their subjectivist interpretation of Dedekind's remarks about "thought", "abstraction", "free creation", etc., miss. It also reveals another respect in which his approach is superior, according to Cassirer's assessment.

Dedekind's talk of "free creation", in particular, is taken by his critics to imply that numbers exist as "mental entities" for him, i.e., in the subjective consciousness of people thinking about them. Cassirer rejects such a reading, as just noted.[19] Nor does he accept, however, that numbers exist "out there" in some crude realist sense. For him, both of those options misrepresent modern mathematics. What matters instead is "complete logical determinateness" (Cassirer 1907, 49), which he understands in a sense tied to mathematical methodology. In the case of introducing the real numbers by means of cuts, Cassirer clarifies this point as follows:

> The "existence" of an irrational number in Dedekind's sense is not intended to mean more than such determinateness: its "being" consists simply in its function of marking a possible division of the realm of rational numbers and thus of a "*position*." (Cassirer 1907, 49 n. 26, my trans.)

[18] As discussed in the essay by Jeremy Heis in the present volume, Russell made other noteworthy contributions to the rise of 20th-century structuralism, however.
[19] For more on Cassirer's defense of Dedekind against the psychologism charge, cf. Yap .(2017)

In Cassirer's 1910 book, the point is explained further:

> The "things" referred to in this treatment are not posited as independent existences [*selbständige Existenzen*] present prior to any relation, but they gain their whole being [*Bestand*], insofar as it is of any concern for the arithmetician, first in and with the relations predicated of them. (Cassirer [1910] 1923, 36, trans. modified)

Similarly two pages later in the same text:

> The whole "being" of numbers rests, along these lines, upon the relations which they display within themselves, and not upon any relations to an outer objective reality [*gegenständliche Wirklichkeit*]. They need no foreign "basis" [*Substrat*], but mutually sustain and support each other insofar as the position of each in the system is clearly determined by the others. (38, trans. modified)

Cassirer's reference to what "concerns the arithmetician", i.e., what matters in terms of mathematical methodology, is significant here. So is his rejection of the view that any "outer objective reality" is involved, either mental or physical. Finally, noting that numbers "need no foreign basis, but mutually sustain and support each other" brings out another core aspect of a structuralist position.

For Cassirer, to ask further questions about the "objective reality" of numbers—ontological questions that go beyond their "logical determinateness"—would bring us back to the realist perspective to which he is fundamentally opposed. This has the following consequence: While the structuralist conception of mathematical objects that Cassirer attributes to Dedekind, and that he accepts himself, amounts to a non-eliminative position, it is not a realist position (in any traditional metaphysical sense); nor is it a form of subjective idealism, psychologism, or nominalism. Cassirer rejects all of these views explicitly. This distinguishes his approach right away from many current forms of structuralism, where the realism vs. nominalism opposition is central.[20] It also brings us back to the "logical idealism" he adopts instead. To quote the crucial passage one more time:

> Logical idealism starts from an analysis of mathematical "objects" and seeks to apprehend the peculiar determinacy of these objects by explaining them through the peculiarity of the mathematical "method," mathematical

[20] One exception is Charles Parsons's form of structuralism. Like Cassirer, Parsons is careful to separate the "non-eliminative" aspect of his position from any additional "realist" or "anti-realist" aspect. It is no coincidence that Parsons's perspective is also shaped strongly by Kant.

concept formation, and the formulation of its problems. (Cassirer [1929] 1965, 405, trans. modified slightly)

As we saw, the core of Cassirer's "logical idealism" is to account for mathematical "existence," "objects," etc., in terms of their "logical determinateness"; and the latter is tied closely to "mathematical method".. Or to be more precise, it reflects the state of mathematical method at Cassirer's time, after the structuralist transformation of modern mathematics. This remark leads over to some further aspects of his position that deserve renewed attention.

4. Function Concepts, Constructions, and Unfoldings

In this section, three further aspects of Cassirer's "logical idealism" concerning mathematics will be highlighted, each of which goes beyond the current literature on structuralism in a noteworthy way. They involve, respectively, his notion of "function concept" and how it is situated historically; the important role Cassirer assigns to "constructions" in mathematics; and his argument that a structuralist conception constitutes the "unfolding" of "germs" present already in earlier stages of mathematics.

4.1. Function Concepts and Functional Thinking

As we saw, in his early works Cassirer characterizes the core difference between more traditional approaches to mathematical science and the novel structuralist perspective, most clearly represented in Dedekind's works, in terms of the distinction between "substance concepts" and "function concepts". What exactly that distinction amounts to is subtle, as it involves a number of ingredients that are never discussed in a fully clear, unified, and definitive way by him.[21] Nevertheless, some of what matters is clear enough. At its core, the crucial change is switching from an Aristotelian perspective on concept formation to a neo-Kantian perspective, both as understood by Cassirer.[22]

According to the position Cassirer ascribes to Aristotle (somewhat crudely, as one might add), concept formation proceeds as follows: We, as thinking subjects, encounter essentially independent objects in the natural world. We then ignore

[21] Cf. Heis (2014) for a helpful, but admittedly still partial, discussion of this topic. See also Kreis (2010), especially chapters 2–4.

[22] In recent discussions, certain forms of structuralism are described as "Aristotelian," as opposed to "Platonist," including some close to Dedekind. From Cassirer's point of view, the latter is rather problematic; i.e., it misrepresents "Dedekind abstraction" fundamentally.

various "marks" these objects have so as to distill out one or a few others, basically by "focusing on them selectively". A simple example would be to observe a red apple and to form the concept of "redness" simply by ignoring everything else about it. This is an illustration of the "substance concept" perspective, both in terms of the underlying realism and the particular conception of abstraction involved (a conception shared by various empiricist thinkers into the 19th century, e.g., J. S. Mill). "Function concepts", in contrast, should be thought of very differently. Not only do we not start with the assumption of fully formed subjects that are affected by independent objects; we also recognize that concept formation, especially in modern science, always involves a form of "constitution" and Kantian "synthesis". And crucially, the latter is based on a kind of "functional unity".

An illustration particularly relevant for present purposes is the difference between thinking of natural numbers as "multitudes of units" and Dedekind's approach to numbers. Along traditional lines, one assumes that some "heap" of objects is given to us directly. One then forms the idea of a corresponding "multitude of units" by ignoring all the differences between the objects in the heap except their numerical distinctness. We are led to Dedekind's alternative "function concept" once we recognize the following: Underlying any such supposedly basic, immediate procedure is a prior ability of "functionally relating" objects, including identifying and distinguishing them in the first place. But then, what is involved in forming a number cannot be as simple as just sketched; it must involve Kantian "synthesis". In fact, already the differentiation of a series of objects, one distinct from the next, does so.[23] And once we recognize that, we are led to thinking of the whole number series in terms of Dedekind's notion of a simple infinity. Rather than abstracting from the "marks" of a given heap of objects in a "subtractive" sense, the form of abstraction at play is more positive. In involves "logical concentration" on the functionally determined structure, here the natural number structure, just as Dedekind taught us. For Cassirer, this is a paradigmatic example of "functional unity".

As this brief sketch indicates, Dedekind's approach to the natural numbers is crucial for Cassirer not just by providing a novel conception of the natural numbers, but by being a model for something deeper and more general. Actually, Dedekind himself is aware of the depth at issue, as the following passage—quoted prominently and approvingly by Cassirer—indicates:

> If we trace closely what is done in counting a group or collection of things, we are led to consider the ability of the mind to relate things to things, to let one

[23] According to Cassirer's neo-Kantian perspective, basic "synthetic" activities include: identifying and differentiating, relating to one another, naming, etc. (see below for more). Here, as at related places in this essay, I am heavily indebted to conversations with Pierre Keller.

thing correspond to another thing, or to represent one thing by another, an ability without which no thought is possible. (Dedekind 1888, 32)

We can understand this passage better if we relate it to another remark in the same text. As background, consider the following: How should we answer one of the two questions raised by Dedekind in the title of his 1888 essay, namely: "Was . . . sollen die Zahlen?" (What is the nature, or better, the point or role of numbers?) His own answer, formulated in the essay's preface, is this: "[Numbers] serve as a means of apprehending more easily and more sharply the difference of things" (31). As these passages indicate, Dedekind is reflectiong on our very ability to think; and for him that includes identifying and differentiating things, representing some by others, naming them, interrelating them in other ways, etc. The most basic role of numbers is to help us in this task, e.g., by arranging things in series: a first, a second, etc. This idea points right back to the concept of simple infinity. It also leads to Dedekind's answer to the second of his two title questions: "Was sind . . . die Zahlen?" (What are numbers, or what is their nature?) Namely, the natural numbers are the things obtained, via "Dedekind abstraction", from any simple infinity. And when suitably extended, such an approach leads to the negative, rational, real, and complex numbers as well, as illustrated most explicitly by his 1872 essay.

One striking thing about the passages by Dedekind just quoted, and about Cassirer's reception of them, is that the notion of function is made absolutely central. The notion of set is not as central; but it too plays a basic role, for both Dedekind and Cassirer (e.g., with respect to the domains and ranges of functions). Nor is the notion of relation quite as central, although it is again important (e.g., when considering the ordering relation on the rational numbers so as to form cuts). Why exactly is the notion of relation not as primary as that of function? The answer is, as Cassirer remarks briefly, that the idea of relation "can be traced back to the more fundamental idea of 'functionality'" (Cassirer 1907, 43); and likewise for the idea of set. In other words, using sets and relations, as we do in modern logic, involves "thinking functionally" in the end. In Frege's and Russell's new logic, with its emphasis on relations and sets or classes, we are moving toward this insight, but we do not quite reach it yet.

4.2. The Crucial Roles of Set-Theoretic Constructions

As just argued, what lies at the bottom of Dedekind's approach to mathematics, and Cassirer's reception of it, is "functional thinking"; and this is illustrated by the central role the successor function plays for the natural numbers. Nevertheless, Dedekind employs sets in crucial ways too. Cassirer picks up on the latter point by emphasizing the role of constructions in modern mathematics more generally. In fact, with his strong emphasis on set-theoretic constructions Cassirer goes, at least in part, against a distinction made prominent by Hilbert and his followers, namely between the "genetic" and the "axiomatic" method. As often claimed by Hilbertians, mathematics in the late 19th and early 20th centuries involved the switch from a "genetic" to an "axiomatic" approach. For Cassirer such a contrast is spurious, since both sides remain crucial.

Cassirer position in this context can again be illustrated, and justified, in relation to Dedekind's work. Take Dedekind's treatment of the real numbers. It is true that the concept of a continuous ordered field does, in some sense or to some degree, provide the basis for that treatment. Along Hilbertian lines, it is then the axiom system by means of which that concept is defined that becomes crucial. But we should not forget about the construction of the system of cuts on the rational numbers. What is the point of that construction, i.e., which basic role or roles does it play? The first such role, explicitly acknowledged by Dedekind and noted by Hilbert as well, is to establish the (semantic) consistency of the concept of a continuous ordered field, or of the corresponding axiom system. But for both Dedekind and Cassirer there is more. The system of cuts also provides the basis for the "abstraction" by means of which "the real numbers" are introduced. This is the second basicrole of the set-theoretic construction. A third role is this: it is only in terms of the cuts that we know how to operate with the real numbers, as is reflected in the fact that the ordering and the arithmetic operations on "the real numbers" are induced directly by those on the system of cuts.

The fact that set-theoretic constructions, like those of Dedekind cuts, play such crucial roles in modern mathematics has more general implications for Cassirer. Let me mention three of them briefly. First, it is in terms of constructing novel mathematical objects out of older ones that these new objects—including all the "ideal elements" characteristic of 19th-century mathematics—become intelligible and acceptable in the first place. This involves making it possible to operate with, say, the real numbers in terms of rational numbers. More basically, it is how we identify and differentiate them, i.e., it grounds their identity. A closely related second point is this: the constructions at issue establish connections between older and newer parts of mathematics. The newer parts are thus not separate and isolated, but integrated into mathematics as a whole from the start. In fact, it is this integration, or a network of corresponding links, that constitutes the unity of mathematics, as Cassirer notes. A third point concerns less mathematics itself than philosophy. For Cassirer, what the importance of such constructions establishes is that Kant was right with his claim that mathematics involves "the construction of concepts". Admittedly, Kant was focused too narrowly on traditional geometric constructions, while with Dedekind's works we see that it is set-theoretic constructions that are crucial for modern mathematics.[24]

4.3. The Historical Unfolding of Structuralist Aspects

I want to mention one more distinctive feature of both Cassirer's reception of Dedekind and of his own structuralist philosophy of mathematics. His juxtaposition of "substance" and "function concepts", as discussed above, may initially be taken to imply that he conceives of the history of mathematics as involving a radical discontinuity or rupture (the move from "substance" to "function concepts"). But this is not quite right. In fact, Cassirer wants to emphasize a corresponding continuity as well. Moreover, that continuity is not unrelated to some of theroles of constructions just sketched. As he writes:

> The new forms of negative, irrational and transfinite numbers are not added to the number system from without but grow out of the *continuous unfolding* of the fundamental logical function that was effective in the first beginnings of the system. (Cassirer [1910] 1923, 67, emphasis added)

The way in which mathematicians like Dedekind have gone from the natural numbers through the negative, rational, and real numbers all the way to the complex numbers by means of set-theoretic constructions is a main example of the "unfolding" Cassirer has in mind (in Dedekind 1854 already). But his conception of "unfolding", and of the corresponding continuity of mathematics, is both richer and subtler than that. Cassirer never spells out that conception clearly and fully in his writings, he only hints at it (including in unpublished manuscripts, e.g., Cassirer 1999). Here is what I take to be the core point: even very early forms of mathematics contain some "functional" aspects, i.e., aspects of the kind of "functional thinking" sketched previously, albeit not in pure forms yet. These aspects are refined and generalized over time, and they come to the fore in the 19th century, especially in works such as Dedekind's. Still, their "germs" go way back, to rudimentary and rather informal parts of mathematics, in fact even beyond what one would normally consider mathematics today.[25]

[24] For more on this point about Cassirer and Kant, cf. Reck and Keller (forthcoming).

[25] For Cassirer, ordinary and mystic ways of thinking are included here, e.g., in terms of the use of number words in magic (where other aspects overshadow the functional/structuralist ones, although they are present in very rudimentary ways). This is one way in which the various "symbolic forms," highlighted in his later writings, are interrelated and build on each other.

A basic illustration of this phenomenon is the following (cf. Heis 2017): consider the natural numbers in the traditional way, i.e., as involving "multitudes of units". Now think of adding two such numbers, e.g., 5 and 7. We can conceive of this as involving three steps: first we count five units, labeled by "1", "2", . . ., "5" ; then we add seven further units, labeled "6", "7", . . ., "12"; finally we record where this leads us, namely to the number 12. Note now that in the second step we treated the sixth unit "as a new 1", by bringing to bear its "position" in the number series. That is to say, we started to reiterate the successor operation with it (the relevant number of times). What we did, in other words, is to utilize an initial segment of the number series and its systematic, step-by-step extension. While obscured somewhat by thinking of numbers as "multitudes of units", this indicates that certain of the aspects distilled out by Dedekind are already at play in this context.

Cassirer's general point here is this: while often mixed together with more traditional and "impure" aspects—geometric, broadly intuitive, also sometimes formalist aspects— in earlier phases of mathematics, "functional" or structuralist aspects can be discerned in all of mathematics, even going back beyond Euclid. Once again, this establishes a unity or continuity for its historical development, across the supposed "substance" vs. "function concept" divide. Put differently, it is what allows us to speak of "mathematics" as one discipline, with a history from at least the ancient Greeks to Cassirer's time. By embedding it in this broad historical panorama, Cassirer has provided a rich historical background and motivation for structuralism in mathematics.

5. Summary and Concluding Remarks

This essay focused on Cassirer's reception of Dedekind's work, which he took to be paradigmatic for a shift from "substance" to "function concepts" in the mathematical sciences. With his sympathetic response to Dedekind's contributions, including defending his remarks about "abstraction" and "free creation", Cassirer went against the mostly critical, often dismissive reactions by other philosophers, both during his time and later, as the examples of Frege, Russell, and Dummett illustrated. And with his characterization of mathematical objects in terms of the notion of "position" in a structure, such as the natural number series, Cassirer anticipated the revival of structuralism in the philosophy of mathematics, by Benacerraf, Resnik, Shapiro, and others 50–70 years later. Both of these facts are remarkable, and a main goal of the present essay was to direct attention to them.

With his positive reception of a Dedekindian structuralism Cassirer did not just anticipate current structuralist positions, however. There are aspects to his approach that are genuinely original and make it distinctive. One example is his discussion—under the umbrella of "logical idealism"—of the specific form of "determinateness" operative in modern mathematics, which is closely related to his rejection of both realist and psychologistic views by him. Three other examples, discussed later in this essay, are the way in which Cassirer emphasizes, with Dedekind, the fundamental role of functional thinking; his emphasis on the roles played by constructions along Dedekindian lines; and the point that the historical development of mathematics, even across the substance/function divide, involves continuity in terms of the "unfolding" of structuralist "germs".

Overall, what Cassirer provides is a treatment of the structuralist transformation of modern mathematics that illuminates not only its logical and metaphysical aspects, but embeds it in a rich developmental and historical story. My summary of it could be enriched further by also covering his reflections on parallel developments in geometry. In the present essay, the focus was exclusively on the side of arithmetic. Both sides led Cassirer to essentially the same conclusions, however.[26] It should be acknowledged, finally, that there are limitations to Cassirer's discussion of structuralism in mathematics too, thus ways in which the current debates go beyond it. For example, he contributed little to its technical development, in the sense that he provided no formal reconstructions of core concepts and proved no new mathematical theorems. After all, he was not a mathematical logician. Then again, with respect to the philosophical and historical dimensions, his treatment deserves to be reconsidered today..

Acknowledgments

This essay has been in the making for over 10 years. An early version was presented at HOPOS 2008, in Vancouver, Canada; subsequent versions at the University of Turin, Italy, in 2010; at the University of Paris 7, Laboratoire SPHERE, France, in 2013; and at the University of Vienna, Austria, in 2015. I would like to thank all the audience members at these events for their comments, and their organizers for giving me a chance to present the material. Beyond that, I am indebted to Jeremy Heis for many conversations about Cassirer's philosophy of mathematics and, even more, to Pierre Keller, who pointed me toward Cassirer in the first place and who continued to influence my views on him substantially. Finally, I am grateful to Georg Schiemer for comments on the penultimate version of this chapter. All the remaining inaccuracies and mistakes are, as usual, my own.

[26] Cf. again Heis (2011), Biagioli (2016), and Schiemer (2018), among others.

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