Value in Stress: A Coherent Approach to Stress-Testing

GLORIA GONZÁLEZ-RIVERA

GLORIA GONZÁLEZ-RIVERA
is an associate professor of economics at the University of California, Riverside.
gloria.gonzalez@ucr.edu

As banking and non-banking financial institutions become increasingly more sophisticated in their trading activities, regulators and risk managers alike need instruments to monitor the solvency and financial stability of their institutions. From a regulatory point of view, financial firms have to maintain minimum capital requirements that are periodically monitored by the regulator. The assessment of capital adequacy is made under two types of standards: 1) those of the 1988 Basle Accord and subsequent amendments, based on minimum risk-weighted capital ratios and value at risk calculations; and 2) those that rely on stress-testing procedures.

Capital requirements for U.S. banking and thrift institutions primarily follow the normative of the Basle Accord. Institutions should maintain a minimum risk-weighted capital ratio. Capital is classified in three tiers, and assets are weighted according to their credit and market risks. At present, the minimum total risk-weighted capital ratio must be 8% of total risk-weighted assets.

The Basle Accord endorses a value at risk (VaR) methodology to assess and monitor market risk capital requirements. The Accord and its subsequent amendments underscore the importance of periodic stress-testing, but the regulation does not provide any specific rule or procedure on how to operationalize these stress tests.

Capital adequacy regulations for U.S. non-banking financial institutions, such as insurance companies and government-sponsored enterprises (GSEs), focus primarily on the ability of these institutions to withstand very stressful scenarios that may create extraordinary losses. The Office of Federal Housing Enterprise Oversight (OFHEO) is the regulator for the GSEs, the Federal National Mortgage Association (FNMA, Fannie Mae) and the Federal Home Loan Mortgage Corporation (FHLMC, Freddie Mac), which operate in the secondary mortgage market. As of the end of 2001, Fannie Mae and Freddie Mac had a combined asset value of more than $1,000 billion.

Unlike the stress-testing guidelines of the Basle Accord, OFHEO's regulation is very specific on formulating the risk-based capital rule. It specifies the stressful scenarios to be considered, the valuation models to use, and the horizon to be considered, among other specifics. GSEs should have sufficient capital to survive at least ten years of stressful economic conditions. The regulation contemplates stressful scenarios that are marked by sharp changes in interest rates (market risk) and house prices (credit risk).

The OFHEO's regulation considers two scenarios. In the up-rate scenario, the maximum increase in the ten-year constant-maturity Treasury is limited to 75% of the average yield over the nine months preceding the stress period. In the down-rate scenario, the maximum decline in the ten-year Treasury rate is limited to 50% of the average yield over the nine months preceding the stress period. It is assumed that interest rate changes all occur in the first year of the stress period and remain at the new level for the remaining nine years. The credit risk component of the stress test assumes that house prices decline according to the historical experience of the early 1980s in the ALMO region (Arkansas, Louisiana, Mississippi, and Oklahoma), where default activity caused losses to creditors exceeding 50% of the collateral assets.

The Committee on the Global Financial System (CGFS) at the Bank for International Settlements (BIS) in 2000 and 2001 produced two reports on the current practices of stress-testing in large financial institutions. The most common practice is to develop a set of historical or hypothetical scenarios and to track their impact on the portfolio of the institution.

The CGFS surveyed 43 commercial and investment banks in ten countries. The banks submitted 293 stress test scenarios such as stock market crashes, and 131 sensitivity stress tests such as changes in correlations or shifts in the yield curve. Most of the stress tests consider extreme movements in equity prices, interest rates, emerging markets, and credit spreads.

The CGFS reports find that even though banks may consider the same scenario, such as a stock market crash, the extent of the crash could be very different across different institutions. Equally heterogeneous is the response and commitment of risk managers to the results of stress-testing. According to the CGFS survey, stress-testing results may help to set limits on position-taking, to allocate capital, to hedge positions, and to challenge modeling assumptions.

In general, stress-testing plays a complementary role in risk management practices of financial institutions. VaR seems to be the dominant methodology. VaR calculations have become a routine exercise for risk managers, and banks and regulators are committed to act upon VaR results. Stress-testing, however, is vaguely defined, and when it is defined, as in the OFHEO's regulation or in the CGFS report, the definition is rather specific to the institution. While there is an extensive professional and academic literature on VaR, stress-testing has not attracted as much interest among academicians, although practitioners and regulators have been paying more attention in recent years, as the CGFS reports attest.

The concept of stress-testing *per se* is very straightforward, but the specification, implementation, and interpretation of the tests are difficult. As the current practice goes, stress-testing is subjective and lacks scientific foundation. Jorion writes

> [Stress-testing] consists of subjectively specifying scenarios of interest to assess possible changes in the value of the portfolio.... Stress testing, however, is poorly adapted to measuring VAR in the same scientific sense as other methods. The method is completely subjective.... Stress testing does not specify the likelihood of worst-case situations [1997, p. 196-198].

The CGFS survey summarizes the difficulties of the current practice when it enumerates at least five limitations of stress-testing:

1. What risk factors to stress.
2. How to combine the stress factors.
3. What range of values to consider.
4. What time frame to analyze.
5. How much is likely to be lost.

We argue for a conceptual shift in our understanding of stress-testing, and show that the limitations of the current practice are due in part to the lack of a methodological framework that can guide the construction of the stress tests.

We propose a rational approach to stress-testing. We present a coherent measure of risk, named *value in stress* (ViS), that answers most of the limitations of stress-testing as expressed by the CGFS reports. At a minimum, we assume a financial institution is aware of the risk factors to which it is exposed, and of the desired time horizon to assess risk. We thus offer answers to limitations 2, 3, and 5.

Notwithstanding the heterogeneity of practices surveyed by the CGFS, there is a common denominator to the current stress-testing in banking institutions and in the OFHEO setting; that is, the stress scenarios are exogenous. *A priori,* the risk manager chooses a set of risk factors to be stressed and the degree of the stress, regardless of the likelihood of the event (although we presume that is small) and of their dependence on the other non-stressed risk factors.

This partial stress practice relates to limitation 1. If the risk factors were independent, partial stress would be acceptable, because on choosing a subset of risk factors we should not expect any reaction in the remaining ones. To overcome limitation 1, we need not only a total stress approach but also a methodology to remove manager subjectivity in choice of the extent of the stress.

Limitations 2 and 3 are consequences, once again, of the exogeneity of the stress scenarios. We argue that considering stressful scenarios fixed as in the OFHEO regulation is not an optimal practice. As corporations gain more experience in managing risk, what is considered stressful should change over time, making fixed stressful scenarios obsolete.

The main characteristic of the proposed ViS measure is that scenarios become endogenous; scenarios will change, depending on the management experience and the level of business of the corporation. Calculation of ViS entails an optimal combination of risk factors and their magnitude, overcoming limitations 2 and 3. Finally, calculation of the ViS permits an assessment of the likelihood of the potential loss, overcoming limitation 5.

I. VALUE IN STRESS

We first define value in stress, and then show it is a coherent measure of risk.

Definition and Properties

Consider an institution with an objective variable V, for instance, net revenue or capital. Assume a time interval $[0, T]$ where the initial position is V_0, and the final position is V_T, so that $\Delta V = V_T - V_0$. Let $f = \{f_1, f_2, ..., f_k\}'$ be a vector of risk factors to which the corporation is exposed. The vector of factor changes, Δf, has a joint probability density function (pdf) $P(\Delta f, \theta)$, where θ is a parameter vector that defines the density function.

From the pdf, we can derive the α probability contour, defined in the space of Δf as the k-dimensional ellipsoid $p(\Delta f, \theta) = c_\alpha$ that includes $100\alpha\%$ probability. The constant c_α delivers the α probability.

Suppose there is a function $h(.)$, continuous and at least twice-differentiable, so that $\Delta V = h(\Delta f)$. That is, only movements in the risk factors drive a change in the objective variable.

The institution wants to assess its risk exposure to extreme changes in the risk factors. For that, the firm chooses an α probability and optimizes its objective function subject to the α probability contour of the set of risk factors.

For simplicity of exposition, let us assume that V represents net revenues, so that the corporation wishes to assess its maximum risk exposure (minimum net revenues), given the α probability contour (extreme events or worst case scenarios). In other words, the corporation searches for the α probability scenarios that produce a maximum loss.*

The problem is stated as

$$\begin{aligned} \Delta V^* \;\equiv\; & \min_{\Delta f} \Delta V = h(\Delta f) \\ & s.t. \quad p(\Delta f, \theta) = c_\alpha \end{aligned} \tag{1}$$

Definition. *Value in stress (ViS) is (minus) the value that minimizes the change in the objective function of the firm, subject to the occurrence of extreme changes in the risk factors:*

$$ViS(\Delta V) = -\Delta V^*$$

ViS is a risk measure. For example, suppose that ΔV^* is negative; that is, stressful movements in the risk factors produce losses. ViS will be positive, and interpreted as the minimum amount that the company needs to add to ΔV in order to avoid negative revenues (or the amount of capital that is needed to withstand these losses); that is, $V_T + ViS = V_0$. If ΔV^* is positive, ViS will be negative, and interpreted as the minimum amount that the company may withdraw from ΔV and still be solvent; that is, $V_T - ViS = V_0$.

Exhibit 1 is a probability contour $p(\Delta f, \theta) = c_\alpha$ for two generic risk factors $(\Delta f_1, \Delta f_2)$. The function $h(\Delta f)$ is assumed to be linear, and iso-value lines can be plotted in the plane $(\Delta f_1, \Delta f_2)$. The ViS value is the tangency point between the ellipsoid and the linear objective function.

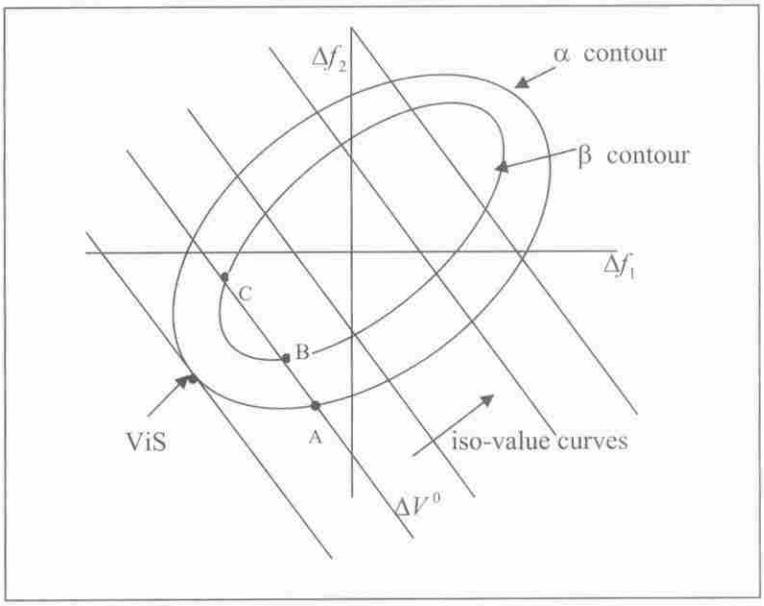
Theorem. *ViS is a coherent measure of risk in the sense of Artzner et al. (ADEH) [1999].*

Proof. For ViS to be coherent, it needs to satisfy the axioms of translation invariance, subadditivity, positive homogeneity, and monotonicity (Definition 2.4 in ADEH):

- *Translation invariance.* Adding a riskless amount to ΔV reduces ViS: $\forall\omega$, $ViS(\Delta V + \omega) = ViS(\Delta V) - \omega$.

  Suppose we add a riskless quantity ω, which does not depend on Δf, to our all final positions so that we need to minimize $\Delta V + \omega$. Then argmin ΔV = argmin$(\Delta V + \omega) = \Delta f^*$. Since $\Delta V^* = h(\Delta f^*)$, and $\min(\Delta V + \omega) = \Delta V^* + \omega$, it follows that $ViS(\Delta V + \omega) = -\Delta V^* - \omega = ViS(\Delta V) - \omega$.
- *Subadditivity.* The sum of two or more different positions is not riskier than the sum of their separate risks: $\forall \Delta V_1, \Delta V_2$, $ViS(\Delta V_1 + \Delta V_2) \leq ViS(\Delta V_1) + ViS(\Delta V_2)$.

EXHIBIT 1
Graphical Representation of Value in Stress



A priori chosen stressful scenarios may not be optimal choices to monitor risk.

Let $\Delta V = \Delta V_1 + \Delta V_2$, $\Delta V_1 = h_1(\Delta f)$, and $\Delta V_2 = h_2(\Delta f)$. Then $\Delta V^* = h(\Delta f^*) = h_1(\Delta f^*) + h_2(\Delta f^*) = \Delta V_1 + \Delta V_2 \geqslant \min \Delta V_1 + \min \Delta V_2 = \Delta V_1^* + \Delta V_2^*$. It follows that $-\Delta V^* \leq -\Delta V_1^* - \Delta V_2^*$, and subadditivity is satisfied.

- *Positive homogeneity.* There are no economies of scale to the size of the position: $\forall \lambda \geqslant 0$, $ViS(\lambda \Delta V) = \lambda ViS(\Delta V)$.

  We have that argmin $\lambda \Delta V$ = argmin ΔV. Since $\Delta V^* = h(\Delta f^*)$ and $\min(\lambda \Delta V) = \lambda \Delta V^*$, it follows that $ViS(\lambda \Delta V) = -\lambda \Delta V^* = \lambda ViS(\Delta V)$.
- *Monotonicity.* If $\forall \Delta V_1$, ΔV_2, and $\forall \Delta f$ such that $\Delta V_1 \leq \Delta V_2$, then $ViS(\Delta V_1) \geqslant ViS(\Delta V_2)$.

  If $\Delta V_1 \leq \Delta V_2$, then $\Delta V_1^* = \min \Delta V_1 \leq \Delta V_2^* = \min \Delta V_2$. It follows that $-\Delta V_1^* \geqslant -\Delta V_2$ and $ViS(\Delta V_1) \geqslant ViS(\Delta V_2)$.

Components of the Definition

Our comments on the main components in the ViS definition relate to the objective function, the endogeneity of stressful scenarios, sensitivity analysis, and diversification.

Objective function. The objective function $h(.)$ is general enough to accommodate a linear or a non-linear relation between the risk factors and the objective variable ΔV. It can be understood as an *aggregated* pricing rule.

Corporations often maintain many business units and many pricing models. There are individual pricing models for individual contracts or financial instruments that are known to the trading parties, but when the corporation as a whole needs to evaluate its total risk exposure, there is no formula. It is not known how to combine individual risk exposures in a meaningful manner. The role of the function $h(.)$ is to summarize all the individual decisions and choices across products and across units, a function that is unknown to the risk managers.

Discovery of the function $h(.)$ needs to rely on econometric methods. From the joint probability distribution of the risk factors, we draw N events of the vector Δf_j, $j = 1, ..., N$. For each event, individual pricing models and information systems are in place to evaluate every contract and every position of the corporation that eventually will generate the corresponding ΔV_j. The risk manager faces a data set $\{\Delta f_j, \Delta V_j\}$ for $j = 1, ..., N$ from which an econometric model can be inferred as:

$$\Delta V_j = h(\Delta f_j) + u_j \tag{2}$$

where u_j is an error term with mean zero and variance σ_u^2.

In this context, ViS is based effectively on minimization of the conditional expectation $E(\Delta V | \Delta f) = h(\Delta f)$. Different parametric and non-parametric methods are readily available to uncover the functional form of this conditional expectation.

It should be noted that the variability of ΔV is due exclusively to the variability of Δf, and consequently we should expect a small error variance σ_u^2. There is no potential misspecification due to the choice of regressors in the econometric model. The probability distribution function of ΔV depends on the distributional assumptions of Δf. Note that if the factors are assumed multivariate-normally distributed, ΔV may not be a normal variate because the function $h(.)$ may be non-linear.

Many risk management methodologies rely on a second-order Taylor expansion of the $h(.)$ function. Our approach takes a different direction because it aims to identify the functional form of $h(.)$ using econometric methods.

Stressful scenarios. The definition of ViS entails that stressful scenarios are endogenous. Once the ViS is calculated, the risk manager knows not only the most stressful value that can be attained within the α probability contour but also the degree of the change in *all* risk factors that needs to occur. This is the main difference between ViS and the current practice in stress-testing.

The design of the OFHEO regulation and the current practice surveyed by the CGFS may result in incomplete measures to monitor risk for several reasons. First, stressing a subset of factors may generate changes in the remaining factors if these are not independent.

Suppose that the vector of factors can be partitioned as $\{\Delta f^{(1)} | \Delta f^{(2)}\}$ and that $\Delta V = h_1(\Delta f^{(1)}) + h_2(\Delta f^{(2)}) + u$. Then $E(\Delta V | \Delta f) = h_1(\Delta f^{(1)}) + h_2(\Delta f^{(2)})$. If the corporation contemplates a shock to $\Delta f^{(1)} = \Delta f^{(1)}_{stress}$, $E(\Delta V | \Delta f^{(1)}_{stress}) = h_1(\Delta f^{(1)}_{stress}) + E(h_2(\Delta f^{(2)}) | \Delta f^{(1)}_{stress})$. The current practice ignores the term $E(h_2(\Delta f^{(2)}) | \Delta f^{(1)}_{stress})$ entirely; consequently, the estimated conditional stress value can be either overestimated or underestimated depending upon the sign of that term. The ViS measure, however, takes into account the interdependence among the factors, and whenever one factor is stressed, there may be a response in the remaining factors depending on the probability model that summarizes the behavior of risk factors, and the functional form of $h(.)$.

Kupiec [1998] argues a similar point with regard to a linear value at risk model. He does not argue for endogenous scenarios, however, but frames his argument following the current practice in stress-testing.

Second, and most important, on choosing *a priori* very specific stressful scenarios, there is no guarantee those will produce the largest losses, or that we can exclude the possibility that the same loss can be achieved under more likely scenarios. Exhibit 1 illustrates this argument.

Consider two factors f_1 and f_2, and assume that the function $h(.)$ is linear. We consider two probability contours. The α contour accounts for a broader range of scenarios than the β contour, and they are less likely to occur than the scenarios in the β contour.

We consider a fixed value of the objective variable, say, $E(\Delta V | \Delta f) = \Delta V^0$. We can construct *iso-value curves* defined as the loci of points $(\Delta f_1, \Delta f_2)$ so that the value ΔV^0 is attained. Points A, B, and C represent different combinations of the two factors, all of which produce the same loss ΔV^0.

Let us suppose the risk manager (or the regulator) has selected the stressful scenario A, which lies on the less likely set of scenarios represented by the α contour. Is A an optimal choice of stressful scenarios? Clearly not, because the point marked ViS, which considers stressful scenarios within the same probability contour as those in A, produces higher losses. Furthermore, the loss ΔV^0 can also occur in the scenarios represented by B and C, which are less stressful than scenario A. Hence, the risk manager will be misled into thinking that the corporation is protected if capital allocations are based on the potential losses ΔV^0. We can find another set of scenarios, as likely to occur as the point A, that delivers greater losses than ΔV^0.

The solution is to let the choice of scenarios be endogenous, because scenarios will change according to the ability and the experience of the corporation to deal with risk, which in turn will affect the functional form of $h(.)$. The OFHEO legislation is a prime example of fixed scenarios.

The current practice does not attach a probability statement to the stress loss; there is merely a perception that the potential loss is unlikely because it is based on unlikely events. This perception can be misleading because $h(.)$ may be non-linear, and stressful scenarios may be associated with likely losses. Exhibit 1 conceptualizes one argument against this perception.

It is relatively simple to calculate the probability of occurrence of the ViS measure, say, p_{ViS}. We need to calculate or to estimate the unconditional probability density function of ΔV, say, $g(\Delta V)$. This is easy to obtain because we have already generated the data on ΔV_j for $j = 1, ..., N$, and we can apply either a kernel estimator or calculate the order statistics.

The probability associated with the ViS measure is $p_{ViS} = \int_{-\infty}^{-ViS} g(\Delta V) d\Delta V$, which can be estimated by calculating the empirical quantiles of ΔV.

Sensitivity analysis. The ViS measure can be complemented with sensitivity analysis. It is important to understand the response of the optimal values Δf^*, ΔV^* to changes in the parameters of the model. For instance, the risk manager may be interested in the sensitivity of ViS to changes in the likelihood of the scenarios, or to changes in the correlation between factors, or to changes in the volatility of some of the risk factors.

The derivatives $\partial\Delta f^*/\partial c_\alpha$, $\partial\Delta f^*/\partial\theta$, $\partial\Delta V^*/\partial c_\alpha$, and $\partial\Delta V^*/\partial\theta$, where θ and c_α are parameters embedded in $p(\Delta f, \theta) = c_\alpha$, can be incorporated in the analysis of stress-testing. We compute these derivatives assuming the risk factors are multivariate-normal in the following section.

ViS calculations are also sensitive to changes in internal models such as prepayment and default models, or evaluation formulas such as derivatives pricing. The function $h(.)$ can be a tool for the risk manager to assess the sensitivity of ViS to changes in internal models.

Diversification. ViS can be implemented across the different business units of the corporation. The subadditivity property guarantees there is risk diversification across units. We can construct different measures of diversification.

For instance, consider a corporation with n units. Since:

$$\begin{aligned} ViS(\Delta V_1 + \Delta V_2 + ...\Delta V_n) &\leq \\ ViS(\Delta V_1) + ViS(\Delta V_2) + ...ViS(\Delta V_n) &\leq \\ n\max\{ViS(\Delta V_1), ViS(\Delta V_2)......ViS(\Delta V_n)\} \end{aligned}$$

a potential diversification measure can be

$$D_{\max} = 1 - \frac{ViS(\Delta V_1 + \Delta V_2 + ...\Delta V_n)/n}{\max\{ViS(\Delta V_1), ViS(\Delta V_2)......ViS(\Delta V_n)\}}$$

where $D_{\max} \geq 0$, provided that $\max\{.\} > 0$.

The quantity $D_{\max}$ means that, for the riskiest unit, diversification produces an average risk reduction of $D_{\max} \times 100\%$. Less risky units will benefit less from diversification (lower D), and some units may not benefit at all (D is non-positive).

As an example, suppose a corporation has two divisions for which $ViS(\Delta V_1) = 30$ and $ViS(\Delta V_2) = 20$, and $ViS(\Delta V_1 + \Delta V_2) = 40$. In absolute terms, diversification produces a reduction in risk of 10. The average value in stress is 20 per division. In this case, Division 2 provides diversification for Division 1. We have that $D_{\max} = 0.33$, so for Division 1, diversification produces a reduction in risk of 33%. For Division 2, $D = 0$, and there is no benefit from diversification for this division.

II. VALUE IN STRESS UNDER NORMALITY

We offer analytical and graphical solutions to the problem stated in Equation (1) under the assumption that the risk factors are multivariate-normally distributed, and we provide a framework where stress-testing can be understood as sensitivity analysis in the ViS framework.

Analytical Solution

Without loss of generality, let us assume we deal with two factors. For instance, the OFHEO regulation contemplates housing prices and interest rates as the two main factors that affect the business of the GSEs. The factors Δf_1 and Δf_2 have means μ_1, μ_2 and variances σ_1^2, σ_2^2, respectively, and they may be correlated with correlation coefficient ρ. If Δf_1 and Δf_2 are bivariate-normal, their joint probability density function is:

$$\begin{aligned} P(\Delta f_1, \Delta f_2) &= \\ \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} &\times \exp\left[-\frac{1}{2(1-\rho^2)}\times\right. \\ \left\{\frac{(\Delta f_1-\mu_1)^2}{\sigma_1^2}\right. &+ \left.\left.\frac{(\Delta f_2-\mu_2)^2}{\sigma_2^2} - \frac{2\rho(\Delta f_1-\mu_1)(\Delta f_2-\mu_2)}{\sigma_1\sigma_2}\right\}\right] \end{aligned}$$

For this distribution, the quadratic form

$$\left\{\frac{1}{1-\rho^2}\left(\frac{(\Delta f_1-\mu_1)^2}{\sigma_1^2} + \frac{(\Delta f_2-\mu_2)^2}{\sigma_2^2} - \frac{2\rho(\Delta f_1-\mu_1)(\Delta f_2-\mu_2)}{\sigma_1\sigma_2}\right)\right\}$$

has a chi-squared distribution with two degrees of freedom. This result permits the construction of elliptical contours.

Since the cumulative distribution function of a chi-square with two degrees of freedom is $\Pr(\chi_2^2 \leq x) = 1 - \exp\{-x/2\}$, the ellipse contour including $100\alpha\%$ probability is given by

$$\frac{1}{1-\rho^2}\left[\frac{(\Delta f_1-\mu_1)^2}{\sigma_1^2} + \frac{(\Delta f_2-\mu_2)^2}{\sigma_2^2} - \frac{2\rho(\Delta f_1-\mu_1)(\Delta f_2-\mu_2)}{\sigma_1\sigma_2}\right] = -2\log(1-\alpha) \tag{3}$$

The optimization problem (1) becomes

$$\Delta V^* \equiv \min_{\Delta f} \Delta V = h(\Delta f) \tag{4}$$

subject to $\frac{1}{1-\rho^2} \times$

$$\left[\frac{(\Delta f_1-\mu_1)^2}{\sigma_1^2}+\frac{(\Delta f_2-\mu_2)^2}{\sigma_2^2}-\frac{2\rho(\Delta f_1-\mu_1)(\Delta f_2-\mu_2)}{\sigma_1\sigma_2}\right]=c_\alpha$$

where $c_\alpha \equiv -2\log(1-\alpha)$.

Let us call $z_1 \equiv (\Delta f_1 - \mu_1)/\sigma_1$ and $z_2 \equiv (\Delta f_2 - \mu_2)/\sigma_2$ the standardized risk factors; and $\partial\Delta V/\partial\Delta f_1 \equiv h_1$ and $\partial\Delta V/\partial\Delta f_2 \equiv h_2$ the sensitivity of the objective function to marginal changes in the risk factors. From the first-order conditions of (4), we have that, in the optimum, the relation as follows must hold:

$$\frac{h_1}{h_2}=\frac{z_1-\rho z_2}{z_2-\rho z_1}\times\frac{\sigma_2}{\sigma_1} \tag{5}$$

The ratio $\frac{h_1}{h_2}$ is a marginal rate of substitution between both factors, or the slope of the iso-value curve; and $\frac{z_1-\rho z_2}{z_2-\rho z_1}\times\frac{\sigma_2}{\sigma_1}$ is the net relative allocation of the risk factors in units of standard deviation, or (minus) the slope dz_2/dz_1 of the elliptical contour.

It is interesting to observe that $z_1 - \rho z_2$ is the net value of z_1. This is the projection error if we were to regress z_1 on z_2. In the optimum, the allocation of risk factors is equal to their marginal rate of substitution. If a corporation is particularly sensitive to one of the factors, the value in stress will load, in relative terms, heavily on such a factor.

Note that the relation in Equation (5) is the basis for understanding that stressing only one set of factors is not an optimal practice. In ViS, whenever a factor is stressed, say, z_1, we expect contemporaneous movements in the remaining factors that may counteract or reinforce the stress on z_1. This is why Equation (5) considers the net effect $z_1 - \rho z_2$.

To solve for the optimal values of both risk factors, and eventually to obtain ΔV^*, we need further knowledge of the function $h(.)$. Let us assume the function is linear, $\Delta V = \beta_1\Delta f_1 + \beta_2\Delta f_2$. Then $h_1 = \beta_1$ and $h_2 = \beta_2$.

Under this framework, the optimal solution for the standardized risk factors is:

$$\begin{aligned} z_1^* &= \pm\left(\rho+\frac{\beta_1\sigma_1}{\beta_2\sigma_2}\right)\sqrt{\frac{c_\alpha}{\left(\rho+\frac{\beta_1\sigma_1}{\beta_2\sigma_2}\right)^2+(1-\rho^2)}} \\ z_2^* &= \pm\left(1+\rho\frac{\beta_1\sigma_1}{\beta_2\sigma_2}\right)\sqrt{\frac{c_\alpha}{\left(\rho+\frac{\beta_1\sigma_1}{\beta_2\sigma_2}\right)^2+(1-\rho^2)}} \end{aligned} \tag{6}$$

The values z^* that minimize ΔV depend on the signs of β_1 and β_2. Of the four possible values of z^*, and once the signs of β_1 and β_2 are known, one will provide the minimum value, and another the maximum value of the function ΔV. Graphically, in the plane $(\Delta f_1, \Delta f_2)$, we can draw the iso-value (ΔV^0) curves

$$\Delta f_2 = \frac{\Delta V^0}{\beta_2} - \frac{\beta_1}{\beta_2}\Delta f_1$$

and the elliptic contour (3).

Exhibit 2 shows the graphical solution for ViS and the optimal choice of scenarios when the two risk factors are bivariate-normally distributed. In this graph we assume that $\beta_1 > 0$ and $\beta_2 > 0$, implying the optimal choice of $z_1^* < 0$ and $z_2^* < 0$. The elliptical contour has been drawn according to the parameters $\mu_1 = \mu_2 = 0$, $\sigma_1^2 = \sigma_2^2 = 1$, $\rho = 0.5$, and $\alpha = 99\%$. For this example, we can interpret that the stressful scenarios for this institution are negative changes in both risk factors of about 2.5 standard deviations, and that, within a 99% probability contour, ViS is approximately 16 units.

Stress-Testing as Sensitivity Analysis

The current practice in stress-testing, as surveyed by the CGFS, can be classified as either choosing *a priori* extreme scenarios and observing the change in portfolio value, or performing a sensitivity test that analyzes the effect of one or more shocks to a single factor in portfolio value. Within the ViS framework, stress-testing is the same as analyzing the sensitivity of the ViS measure to changes in the several parameters included in the optimization problem (1).

We classify stress-testing into three categories.

Stress 1. The OFHEO regulation and the current practice in stress-testing fall into the Stress 1 category. Stressful scenarios are chosen according to historical experience, such as the drop in housing prices in the ALMO region at the beginning of the 1980s, or the stock market crash of 1987.

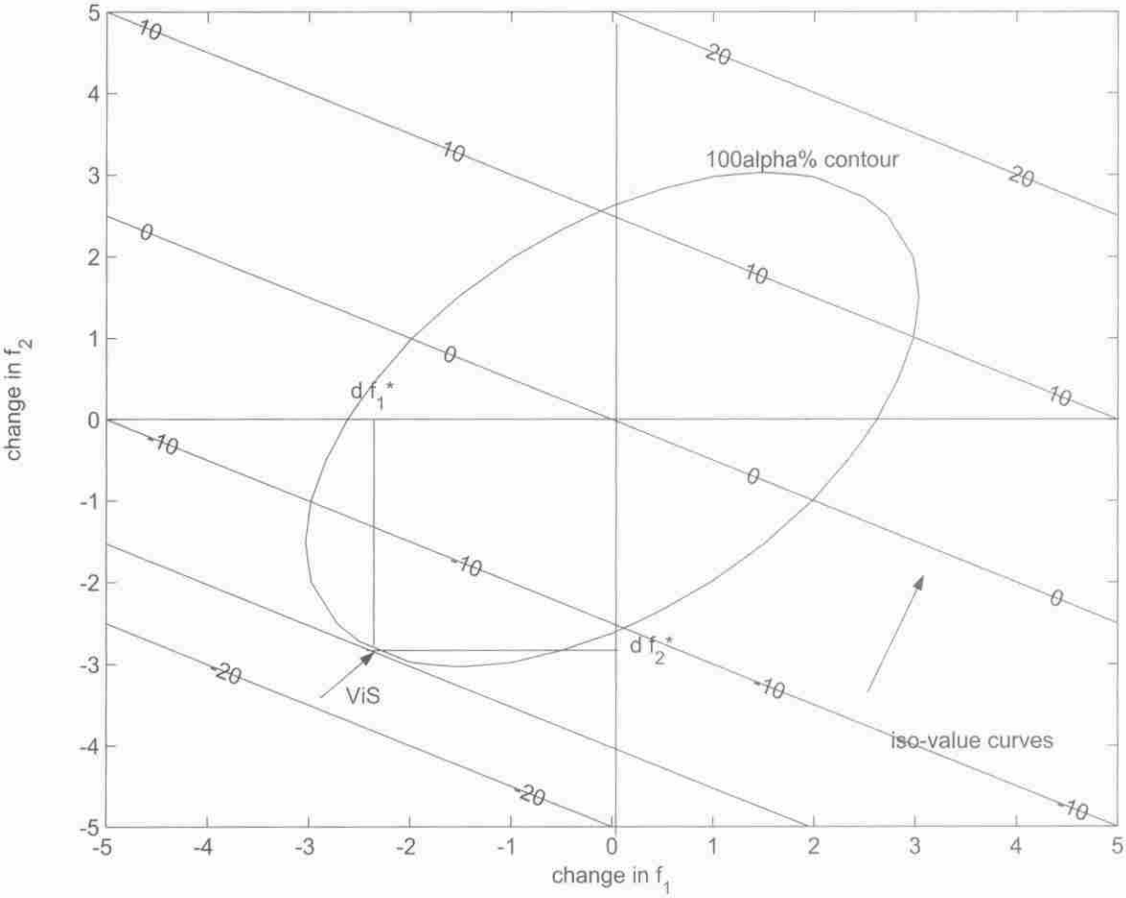
Stress 1 is defined as the collection of scenarios that are considered extreme events within a given probability model; that is:

$$\Delta f^{stress1} = \{\Delta f \mid P(\Delta f, \theta) \ll \varepsilon\}$$

where $P(\Delta f, \theta) \ll \varepsilon\}$ means that the probability of the scenario is very small.

EXHIBIT 2

Value in Stress under Bivariate-Normality



The parameters of the bivariate-normal probability distribution (Δf_1, Δf_2) are $\mu_1 = \mu_2 = 0$, $\sigma_1^2 = \sigma_2^2 = 1$, $\rho = 0.5$. The probability contour is drawn for α = 99%. For the linear function ΔV, it is assumed that $\beta_1 > \beta_2 > 0$.

In the ViS framework, the stress of the scenario is related to the choice of α and consequently the choice of c_α. It is of interest to calculate the sensitivity of the optimal risk factors and of the ViS measure to changes in c_α. For bivariate-normality of the risk factors and linearity of the function ΔV, we have that:

$$\frac{\partial z_1^*}{\partial c_\alpha} = \frac{1}{2c_\alpha} z_1^* < 0$$
$$\frac{\partial z_2^*}{\partial c_\alpha} = \frac{1}{2c_\alpha} z_2^* < 0$$

assuming that min ΔV is achieved when $z_1^* < 0$ and $z_2^* < 0$. The change in ViS follows immediately from $\frac{\partial V^*}{\partial c_\alpha} = \beta_1 \frac{\partial \Delta f_1^*}{\partial c_\alpha} + \beta_2 \frac{\partial \Delta f_2^*}{\partial c_\alpha}$. As expected, if the scenarios are more stressful, the ViS measure increases, everything else equal.

Stress 2. In the same probability model, it is of interest to assess the sensitivity of ViS to changes in the parameters of the model; say, θ becomes $\tilde{\theta}$. For instance, we would like to analyze the expected changes in ViS when the correlation between risk factors strengthens or weakens.

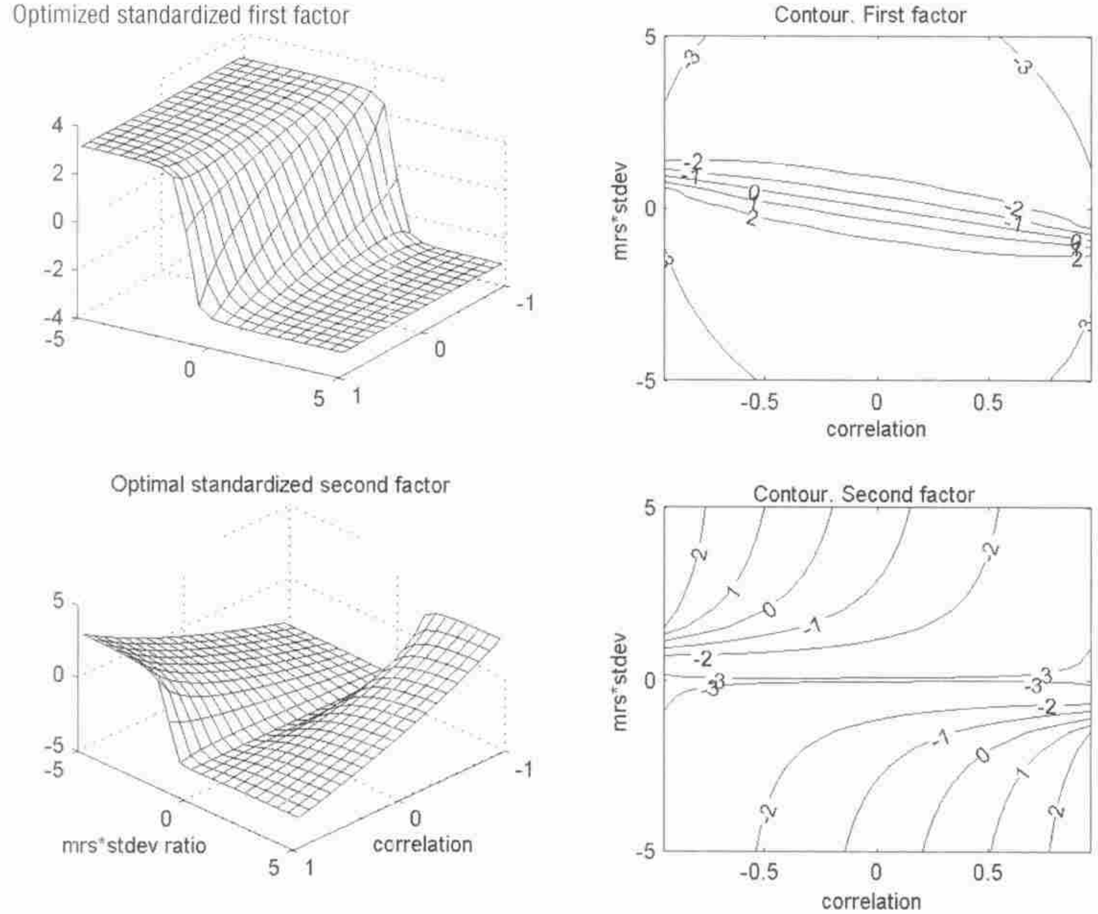
Stress 2 is defined as the collection of stressful scenarios for which the parameters that define the probability model are slightly modified; that is:

$$\Delta f^{stress2} = \{\Delta f \mid (P(\Delta f, \theta) \ll \varepsilon) \rightarrow (P(\Delta f, \tilde{\theta}) \ll \varepsilon)\ \}$$

Under the same assumptions as in Stress 1, we show the changes in the optimal values of the risk factors when there are marginal changes in the correlation coefficient ρ, and in the ratio $\frac{\beta_1 \sigma_1}{\beta_2 \sigma_2}$.

Let us call $D \equiv \left(\rho + \frac{\beta_1 \sigma_1}{\beta_2 \sigma_2}\right)^2 + (1 - \rho^2) > 0$. Then:

EXHIBIT 3
Optimal Standardized Risk Factors



Risk factors as a function of the correlation coefficient and the weighted ratio of standard deviation.

$$\frac{\partial z_1^*}{\partial \rho} = -\frac{1+\rho\frac{\beta_1\sigma_1}{\beta_2\sigma_2}}{D}\sqrt{\frac{c_\alpha}{D}}\begin{cases} <0 & if \quad (1+\rho\frac{\beta_1\sigma_1}{\beta_2\sigma_2})>0 \\ >0 & if \quad (1+\rho\frac{\beta_1\sigma_1}{\beta_2\sigma_2})<0 \end{cases}$$

$$\frac{\partial z_2^*}{\partial \rho} = -\frac{\left(\frac{\beta_1\sigma_1}{\beta_2\sigma_2}\right)^2\left(\rho+\frac{\beta_1\sigma_1}{\beta_2\sigma_2}\right)}{D}\sqrt{\frac{c_\alpha}{D}}$$

$$\begin{cases} <0 & if \quad (\rho+\frac{\beta_1\sigma_1}{\beta_2\sigma_2})>0 \\ >0 & if \quad (\rho+\frac{\beta_1\sigma_1}{\beta_2\sigma_2})<0 \end{cases}$$

$$\frac{\partial z_1^*}{\partial(\frac{\beta_1\sigma_1}{\beta_2\sigma_2})} = -\frac{1-\rho^2}{D}\sqrt{\frac{c_\alpha}{D}}<0$$

$$\frac{\partial z_2^*}{\partial(\frac{\beta_1\sigma_1}{\beta_2\sigma_2})} = \frac{\left(\frac{\beta_1\sigma_1}{\beta_2\sigma_2}\right)\cdot(1-\rho^2)}{D}\sqrt{\frac{c_\alpha}{D}}$$

$$\begin{cases} <0 & if \quad (\beta_1/\beta_2)<0 \\ >0 & if \quad (\beta_1/\beta_2)>0 \end{cases}$$

Exhibit 3 shows the graphical representation of the z_1^* and z_2^* given in Equation (6) as a function of the correlation coefficient and the weighted ratio of standard deviations. The first factor is very responsive to changes in $\frac{\beta_1\sigma_1}{\beta_2\sigma_2}$ when this value is in the neighborhood of zero, but for large values the function becomes quite flat. The sensitivity of the first factor with respect to the correlation coefficient depends on the value of $\frac{\beta_1\sigma_1}{\beta_2\sigma_2}$. When the ratio is low, the first factor is more sensitive to changes in the correlation coefficient than when the ratio is high.

The second factor behaves in the opposite direction. It is more sensitive to changes in $\frac{\beta_1\sigma_1}{\beta_2\sigma_2}$ as well as to changes in the correlation coefficient when the ratio is high.

Stress 3. Suppose a different probability model is assumed. For instance, we might challenge the assumption of the bivariate-normality of the risk factors, and we would like to assess the sensitivity of ViS with respect to a new probability distribution that may entail more correlation between factors in a downturn than in an upturn.

Stress 3 is defined as the collection of stressful scenarios when a different probability model for the risk factors is selected; that is:

$$\Delta f^{stress3} = \{\Delta f \mid (P(\Delta f, \theta) \ll \varepsilon) \to (W(\Delta f, \phi) \ll \varepsilon) \}$$

This is the case we analyze more fully.

III. EXTENSIONS

The assumption of joint normality of the risk factors is convenient because is tractable, and, on specifying the mean and the variance-covariance matrix, we can characterize the complete dependence structure of the risk factors. It is only within the elliptical world that the correlation coefficient is enough to describe the dependence of risk factors. Beyond the elliptical world, factors can be dependent and have zero correlation. In the non-elliptical world, we need to specify the joint probability density of the factors if we wish to capture the dependence structure among risks.

We consider two approaches when we wish to depart from the distributional assumption of normality: non-parametric estimation, and copula functions.

Non-Parametric Estimation of the Joint Density Function

In the case of non-parametric estimation, the density is estimated from the historical behavior of the risk factors. Upon collecting data in the vector of factors Δf, the non-parametric estimator of the joint density is

$$\hat{p}(\Delta f) = \frac{1}{nh^d} \sum_{i=1}^{n} K \left\{ \frac{1}{h} (\Delta f - \Delta f_i) \right\}$$

where n is the number of observations; h is the window width or smoothing parameter that can be chosen optimally $h_{opt} = An^{-1/(d+4)}$; d is the dimension of the vector or number of factors considered; and $K(.)$ is the kernel function defined for the d-dimensional vector satisfying:

$$\int_{\Re^d} K(\Delta f) d\Delta f = 1$$

The kernel function is customarily a radially symmetric unimodal probability function; for example, the normal kernel where

$$K(\Delta f) = (2\pi)^{-d/2} \exp(-\frac{1}{2} \Delta f\,'\Delta f)$$

but there are many other functions the researcher can choose.

For a discussion of different kernel functions and choice of the window width, see Silverman [1986] and Pagan and Ullah [1999]. The non-parametric approach is recommended when there are not a large number of factors, as in the OFHEO regulation.

Copula Functions

Nelsen [1999] refers to copulas as "functions that join or couple multivariate distribution functions to their one-dimensional marginal distribution functions." Suppose we know the marginal distribution function of our risk factors. Let us assume we have two factors Δf_1 and Δf_2 with distribution functions $F_1(\Delta f_1)$ and $F_2(\Delta f_2)$. The Sklar theorem [1959] says that the joint cumulative distribution function can be expressed as

$$F(\Delta f_1, \Delta f_2) = C(F_1(\Delta f_1), F_2(\Delta f_2))$$

where C is the copula function.

In order to obtain the joint distribution, we need to know the marginal probability of each risk and to select the copula function. For instance, suppose marginal distributions are standard-logistic; that is, $F_1 = [1 + \exp(-\Delta f_1)]^{-1}$ and $F_2 = [1 + \exp(-\Delta f_2)]^{-1}$, and we choose the Ali, Mikhail, and Haq copula:

$$C_\theta(u, v) = \frac{uv}{1 - \theta(1-u)(1-v)}$$

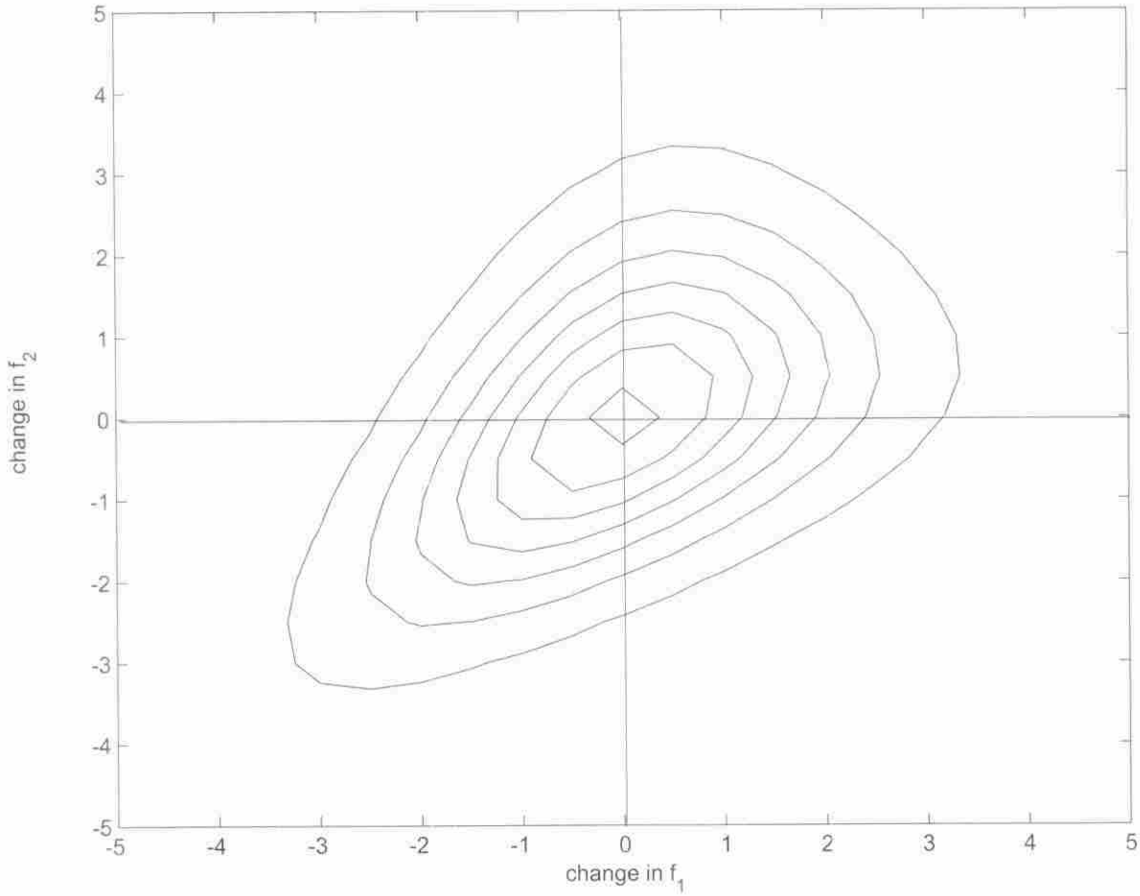
Then the joint distribution function of the risk factors is:

$$\begin{aligned} F(\Delta f_1, \Delta f_2) &= C_\theta(F_1(\Delta f_1), F_2(\Delta f_2)) \\ &= [1 + \exp(-\Delta f_1) + \exp(-\Delta f_2) + \\ &\quad (1-\theta)\exp(-\Delta f_1 - \Delta f_2)]^{-1} \end{aligned}$$

where the parameter $\theta \in [-1, 1]$ guides the degree of dependence between the two factors. For $\theta = 1$, we have the Gumbel bivariate-logistic distribution whose contours are described in Exhibit 4.

In this case, we can see there is a stronger dependence between the two risk factors when their changes are negative than when they are positive.

EXHIBIT 4
Contours of the Gumbel Bivariate Logistic Distribution



Dependence between the two risk factors is stronger when their changes are negative than when they are positive.

IV. CONCLUSIONS

Stress-testing has not attracted as much interest in the academic community as the value at risk methodology, but it has gained a great deal of attention from practitioners and regulators in recent years. Stress-testing is a recommended practice for banking institutions in the Basle Accord and its amendments, and it is a mandatory practice for non-banking institutions like the government-sponsored enterprises that must report to the regulator, the Office of Federal Housing Enterprise Oversight.

We have provided a methodological framework for stress-testing. The coherent measure of risk, value in stress, responds to the limitations of the current practice in stress-testing as expressed by the Committee on the Global Financial System. Basically, the limitations of stress-testing are related to a lack of guidelines on how to choose scenarios and their magnitude, and to the absence of a probability associated with the potential losses.

The heterogeneous current practices across banking institutions, as surveyed by the CGFS, and the OFHEO regulations share one common feature; that is, the stress scenarios are chosen *a priori*. The risk manager chooses a subset of risk factors to be stressed and the degree of the stress, regardless of its likelihood (although we presume it should be low) and regardless of any dependence on the remaining non-stressed factors. This behavior could be misleading, and we have argued that the choice of scenarios should be endogenous.

The set of stress scenarios may vary according to economic conditions and the current solvency of the institution. For instance, the OFHEO regulation considers two stress scenarios, up-rate and down-rate. But the GSEs are extremely skillful in managing market risk and credit risk. They make extensive use of hedging instruments to protect themselves against movements in interest rates in any direction, and they have developed sophisticated tools to control credit risk. Hence, the two rigid scenarios that the OFHEO regulation contemplates, even though stressful, may not be the most relevant to monitor capital adequacy.

Furthermore, the dynamic behavior of institutions can make risks that are barely contemplated by the regulation, such as liquidity risk or counterparty risk, more relevant to their future solvency. These additional risks can be incorporated in the ViS framework, and they can deliver a more complete picture of the capital adequacy of the institutions.

We have shown, under an assumption of the joint normality of the risk factors, that ViS can be easily implemented, and that the current practice in stress-testing may be reduced to sensitivity analysis within the ViS framework. If normality is very restrictive, we have shown that non-parametric estimation and copula functions are two easily implemented avenues.

ENDNOTES

The author thanks Vassilis Lekkas, Donald Solberg, and Carol Wambeke for introducing her to stress-testing. Their insights and discussions were most valuable.

*The same problem can be rephrased in terms of capital. The corporation wishes to allocate enough capital to withstand the losses produced by extreme movements in the risk factors. In this case, it will be interested in assessing worst case losses subject to events that have an extreme probability of occurrence.

REFERENCES

Ali, M., N.N. Mikhail, and M.S. Haq. "A Class of Bivariate Distributions Including the Bivariate Logistic." *Journal of Multivariate Analysis*, 8 (1978), pp. 405-412.

Artzner, P., F. Delbaen, J.M. Eber, and D. Heath. "Coherent Measures of Risk." *Mathematical Finance*, 9 (1999), pp. 203-228.

Jorion, P. *Value at Risk: The New Benchmark for Controlling Market Risk*. New York: McGraw-Hill, 1997.

Kupiec, P.H. "Stress Testing in a Value at Risk Framework." *The Journal of Derivatives*, 6 (1998), pp. 7-24.

Nelsen, R.B. *An Introduction to Copulas*. New York: Springer, 1999.

Pagan, A., and A. Ullah. *Nonparametric Econometrics*. New York: Cambridge University Press, 1999.

Silverman, B.W. *Density Estimation for Statistics and Data Analysis*. New York: Chapman and Hall, 1986.

Sklar, A. "Fonctions de Répartition à n Dimensions et Leurs Marges." Publ. Inst. Statist. Univ. Paris, 8 (1957), pp. 229-231.

"Stress Testing by Large Financial Institutions: Current Practice and Aggregation Issues." Bank for International Settlements, Committee on the Global Financial System, 2000.

"A Survey of Stress Tests and Current Practice at Major Financial Institutions." Bank for International Settlements, Committee on the Global Financial System, 2001.