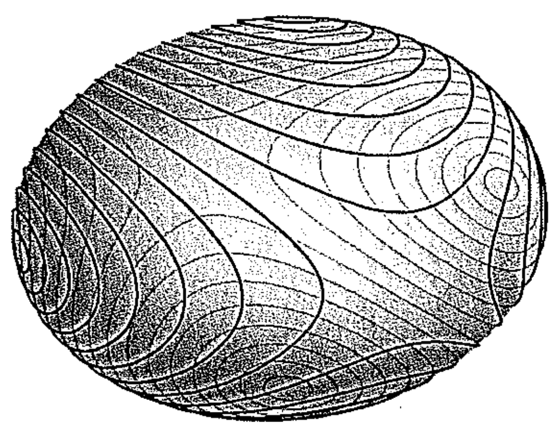
Solutions

Begin the exam when instructed to do so. You have 150 minutes to complete 10 problems. The maximum point credit is shown at the beginning of each problem. All work must be on the exam pages.

The problems have been ordered so that the ones requiring logical thinking of the highest caliber are first, reserving the more mechanical problems which you can do on auto-pilot for the end, after exam fatigue has set in.

Rules: No sources (notes, texts, homework solutions, etc.) or calculators are allowed. Formulas can be found at the end of the exam.

The Cornell Code of Academic Integrity is in effect, as always.



1. (8 points) In torque-free rotational motion of a general rigid body, which of the following are **constant** in time?

   - $\vec{\omega}$
   - space-frame components ω'_α
   - body-frame components ω_α
   - $\vec{L}$
   - space-frame components L'_α
   - body-frame components L_α
   - space-frame components $I'_{\alpha\beta}$
   - body-frame components $I_{\alpha\beta}$

2. (6 points) A **basketball, football** and **tennis racket** are tossed upward and their rotational motion is observed for various initial conditions. Neglecting the effects of air flow and other sources of torque, list the objects (if any)

   (a) that always rotate about an axis whose direction is fixed in space:

   basketball

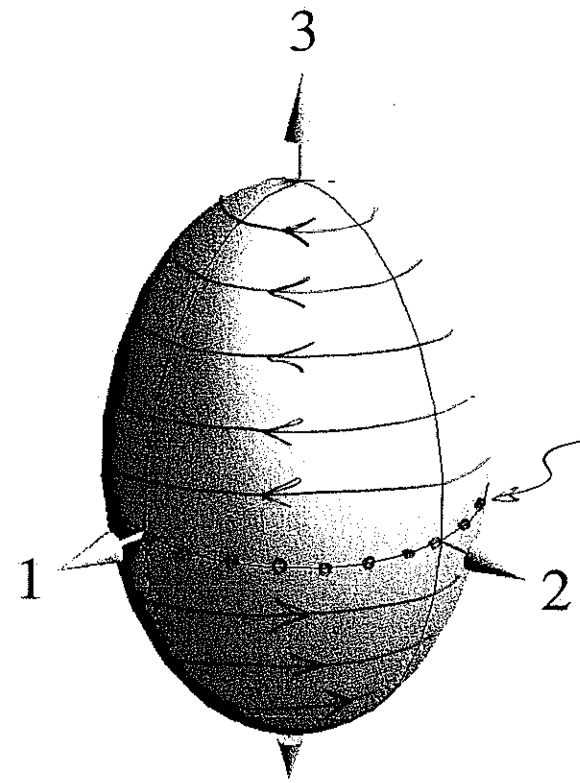
   (b) that always have a body axis that is either static or performs simple precession about a fixed direction in space:

   basketball, football

   (c) for which the body frame components of $\vec{\omega}$ are periodic (or constant) functions of time (aside from very special initial conditions):

   basketball, football, tennis racket

3. (6 points) Below is the drawing of the surface $T_{\text{rot}}(\omega_1, \omega_2, \omega_3) = E_0$ for a **symmetric top** whose body-3-axis is the symmetry axis:



equator is static

(a) Is this top prolate ($I_3 < I_1 = I_2$) or oblate ($I_3 > I_1 = I_2$)?

(b) Draw ω-trajectories on the ellipsoid above, with arrows giving the direction of time, and enough initial conditions to give a complete picture of the possible motions. [Hint: check out the Euler equations at the end of the exam or seek inspiration from the cover sheet.]

$\Omega \propto \omega_3$: switches sign at equator

4. (10 points) You are in a "body" frame rotating with constant angular velocity ω about a fixed axis in space.

   (a) Suppose you are only interested in the effects of the rotating frame on **static equilibrium** properties (particles always at rest in your frame). In this case there is just one fictitious force you need to consider; which one?

   centrifugal force : $-m\vec{\omega}\times(\vec{\omega}\times\vec{r})$

   (b) The effects of the fictitious force can, alternatively, be explained by the fictitious potential

   $$U_{\text{fict}} = -A(x^2+y^2),$$

   where x and y are measured on axes perpendicular to the rotation axis. Determine A in terms of ω and the particle mass m.

   $$\vec{\omega}\times(\vec{\omega}\times\vec{r}) = (\vec{\omega}\cdot\vec{r})\vec{\omega} - \omega^2\vec{r}$$

   $$A = \frac{1}{2}m\omega^2$$

   $$= (\omega z)\omega\hat{z} - \omega^2(x\hat{x}+y\hat{y}+z\hat{z})$$

   $$= -\omega^2(x\hat{x}+y\hat{y})$$

   (c) A rotating planet made of viscous fluid fits the situation described above: all the fluid particles are at rest in the body frame (due to viscous friction) and the fluid planet surface is a surface of constant total potential energy $U_{\text{gravity}} + U_{\text{fict}}$. Suppose the planet is rotating slowly, so the polar radius R is nearly the same as the equatorial radius $R+h$. Make a rough estimate of h in terms of ω, R and the surface gravity of the planet, g. [Hint: consider dimensional analysis.]

   $$-\frac{1}{2}m\omega^2R^2 + mgh \cong 0$$

   $$h = \frac{1}{2}\frac{\omega^2R^2}{g}$$

5. (20 points) An electrical LC-circuit is a series arrangement of a solenoid of inductance L with a parallel-plate capacitor of capacitance C. In the limit of low electrical resistance, the dynamics of the charge $Q(t)$ on one of the capacitor plates is described by the Lagrangian

$$\mathcal{L}(Q,\dot{Q}) = \frac{1}{2}L\dot{Q}^2 - \frac{1}{2}Q^2/C.$$

The current flowing in the circuit is $i = \dot{Q}$.

(a) On the basis of this Lagrangian, is the energy stored in the inductor "kinetic" or "potential" in nature?

(b) Determine the momentum P conjugate to the charge Q and write down the Hamiltonian $H(Q,P)$ associated with the LC-circuit.

$$P = \frac{\partial \mathcal{L}}{\partial \dot{Q}} = L\dot{Q}$$

$$H = \frac{P^2}{2L} + \frac{Q^2}{2C}$$

(c) What are the units of P? [Hint: L has units of tesla-meter^2/ampere, i.e. magnetic flux over current.]

$\dot{Q} = \text{amp}$ $\quad P = L\dot{Q} = \text{tesla-m}^2$

$= \text{magnetic flux}$

(d) Show that the product QP still has units of **action**. [Hint: use the Lorentz force law $\vec{F} = q\vec{v} \times \vec{B}$ to convert electromagnetic to mechanical units.]

$QP = \text{coul} \times \text{tesla} \times \text{m}^2$

$qvB = \text{coul}\ \frac{\text{m}}{\text{s}}\ \text{tesla} = \text{newton} = \text{Joule/m}$

$\Rightarrow \text{coul} \times \text{tesla} \times \text{m}^2 = \text{Joule} \times \text{s} = \underline{\text{action}}$

(e) H is a harmonic oscillator and can be expressed in terms of action-angle variables as $H'(I, \theta) = \omega I$. Determine ω directly from the Lagrangian or the Hamiltonian of part (b).

$$H = \frac{P^2}{2L} + \frac{1}{2} L \omega^2 Q^2 \qquad \omega = \frac{1}{\sqrt{LC}}$$

(f) The capacitance C is changed **adiabatically** (by changing the plate separation) from an initial value to a final value that is twice as large. Either state that the energy of the oscillator is unchanged by this or give the factor by which it is changed. [Hint: consider the Hamiltonian H' of part (e).]

$$H' = \omega I \qquad I \cong \text{const.}$$

$$C \rightarrow 2 \times C$$

$$\omega \rightarrow \omega/\sqrt{2}$$

$$E = H' \rightarrow E/\sqrt{2}$$

energy is reduced by $1/\sqrt{2}$

6. (8 points) Consider a Hamiltonian for an oscillator (1 degree of freedom) which has the following form in action-angle variables:

$$H(I,\theta) = A\sqrt{I}. \qquad E^2 = A^2 I$$

Determine the period T of the oscillator when its energy is E. Express your answer in terms of E and the constant A.

$$\frac{2\pi}{T} = \frac{\partial H}{\partial I} = \frac{1}{2}\frac{H}{I} = \frac{E}{2I} = \frac{E}{2E^2}A^2$$

$$= \frac{1}{2}A^2/E$$

$$T = 4\pi\, E/A^2$$

7. (3 points) Circle the precise statement of "Poincaré recurrence"[1]:

- Any initial point within an arbitrarily small volume V of phase space will, over the course of time, visit V arbitrarily many times.
- Any initial point within an arbitrarily small volume V of phase space will, given enough time, return to V.
- Within an arbitrarily small volume V of phase space there exists an initial point that, given enough time, returns to V.

[1] The accessible phase space for the Hamiltonian system under consideration has finite volume when the energy is bounded.

8. (16 points) The following questions all concern the Hamiltonian[2]

$$H(q,p) = A\log(qp),$$

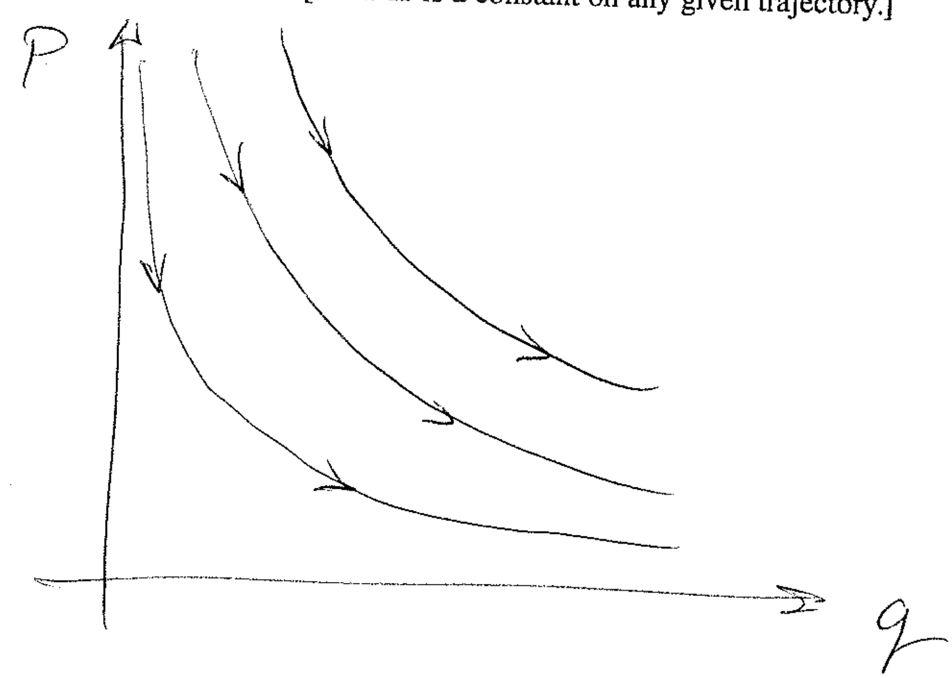
where A is a constant and q, p are the generalized coordinate and conjugate momentum, respectively. You may confine your attention to the quadrant of phase space where **both** q **and** p **are positive**.

(a) Write down Hamilton's equations of motion for H.

$$\dot{q} = A/p$$

$$\dot{p} = -A/q$$

(b) Sketch a few trajectories, showing how they fill phase space. Indicate the direction of flow with arrows. [Hint: H is a constant on any given trajectory.]



[2] $\log(\cdots)$ is the "natural" logarithm function.

(c) H can be simplified with the help of the generating function $F(q, Q) = e^Q \log q$. First express the transformed canonical variables in terms of the original ones:

$Q(q,p) = \log(pq)$ $P(q,p) = -pq \log q$

$$p = \frac{\partial F}{\partial q} = e^Q/q \Rightarrow Q = \log(pq)$$

$$P = -\frac{\partial F}{\partial Q} = -e^Q \log q \Rightarrow P = -(pq)\log q$$

(d) The transformed Hamiltonian is $H'(Q, P) = A\,Q$ (use this to check your canonical transformation). Write down the most general trajectory for H':

$Q(t) = Q_0$ $P(t) = P_0 - At$

$$\dot{Q} = 0$$

$$\dot{P} = -A$$

(e) Using your answer to (d) and the inverse of the canonical transformation (c) determine the most general solution $q(t)$ of the original equations of motion.

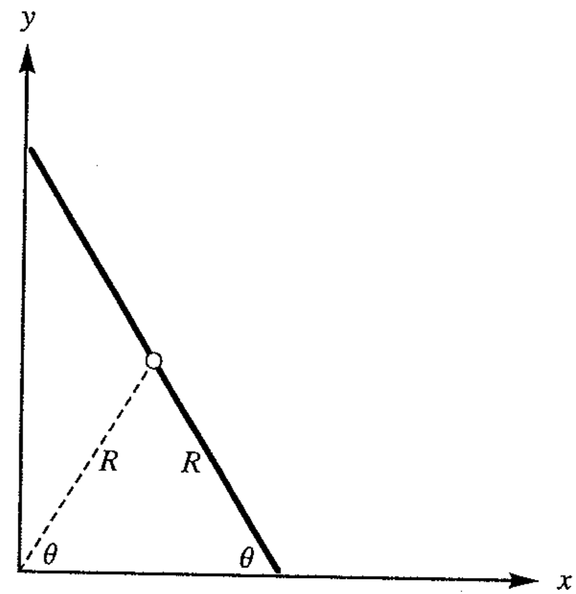
$$\log q = -Pe^{-Q} \qquad q = e^{-Pe^{-Q}}$$

$$q(t) = e^{(At - P_0)e^{-Q_0}}$$

9. (3 points) The slow precession of the major-minor axes of elliptic orbits ("periapsis precession") of planets such as Mercury is the result of (choose one)

   - a small torque applied to the orbiting body by the central body.
   - small departures from the inverse-square law of gravitation caused by non-sphericity of the central body.
   - the natural precession that may occur in any free rotational motion, similar to the wobble of the football.
   - the orbiting body's frame being non-inertial.

10. (20 points) A ladder of mass M and length $2R$ slides against two frictionless surfaces, a vertical wall (y-axis) and a horizontal floor (x-axis):



The center of mass of the ladder is halfway between the ends and is also indicated in the drawing to help you in your calculations. Use the tilt of the ladder, θ, as the generalized coordinate for this one-degree-of-freedom system.

(a) The ladder's kinetic energy is the sum of translational and rotational kinetic energies, of and about the center of mass. Express the rotational kinetic energy T_{rot} in terms of the ladder's moment of inertia I for rotations about the center of mass (and axis perpendicular to the drawing).

$$T_{rot} = \frac{1}{2} I \dot{\theta}^2$$

(b) Write down the ladder's translational kinetic energy, T_{trans}.

$$T_{trans} = \frac{1}{2} M (R\dot{\theta})^2$$

(c) Write down the ladder's gravitational potential energy, V_{grav}.

$$V_{grav} = MgR\sin\Theta$$

(d) Write out the Lagrangian of the system, including all of the above contributions.

$$L = \frac{1}{2}I\dot{\Theta}^2 + \frac{1}{2}M(R\dot{\Theta})^2 - MgR\sin\Theta$$

(e) Write (but don't solve) the Euler-Lagrange equation for this system.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\Theta}}\right) = \frac{\partial L}{\partial \Theta} \;:\; (I + MR^2)\ddot{\Theta} = -MgR\cos\Theta$$

(f) Which of the terms, T_{rot}, T_{trans}, V_{grav}, would change if the wall were moving horizontally?

T_{trans}

Formulas

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0$$

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

$$H = \sum_i p_i \dot{q}_i - L$$

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}$$

$$S = \int_{t_1}^{t_2} L\, dt$$

$$\mathbf{r}_1 = \left(\frac{m_2}{m_1 + m_2}\right)\mathbf{r} \qquad \mathbf{r}_2 = -\left(\frac{m_1}{m_1 + m_2}\right)\mathbf{r}$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

$$A = G m_1 m_2$$

$$L_z = \mu r^2 \dot{\theta}$$

$$\mu \ddot{r} = -\frac{dU}{dr} \qquad U(r) = \frac{L_z^2}{2\mu r^2} - \frac{A}{r}$$

$$r(\theta) = \frac{r_0}{1 + \epsilon \cos\theta}$$

$$r_0 = \frac{L_z^2}{\mu A} \qquad E = \frac{A}{2r_0}(\epsilon^2 - 1)$$

$$b = r_0/\sqrt{\epsilon^2 - 1} \quad (\epsilon > 1)$$

$$a = r_0/(1 - \epsilon^2) \qquad b = \sqrt{1 - \epsilon^2}\, a$$

$$\dot{\mathcal{A}} = \frac{L_z}{2\mu}$$

$$a^3 = \frac{G(m_1 + m_2)}{4\pi^2} T^2$$

$$I = \sum_{i=1}^{N} p_i \left.\frac{dQ_i}{ds}\right|_{s=0} - F$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

$$S = \int_{t_1}^{t_2} \left(\sum_{i=1}^{N} p_i \dot{q}_i - H \right) dt$$

$$\{f, g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial g}{\partial q}\frac{\partial f}{\partial p}$$

$$\{Q, P\} = 1$$

$$\dot{A} = \{A, H\} + \frac{\partial A}{\partial t}$$

$$F = F(q, Q, t) \qquad p = \frac{\partial F}{\partial q} \qquad P = -\frac{\partial F}{\partial Q}$$

$$H'(Q, P, t) = H(q(Q, P, t), p(Q, P, t), t) + \frac{\partial F}{\partial t}$$

$$\frac{\partial H}{\partial I} = \omega = \frac{2\pi}{T}$$

$$\mathbf{U}^{\mathrm{T}}\mathbf{U} = \mathbf{U}\mathbf{U}^{\mathrm{T}} = 1$$

$$\mathbf{r}' = \mathbf{U}\mathbf{r} \qquad \dot{\mathbf{r}}' = \dot{\mathbf{U}}\mathbf{U}^{\mathrm{T}}\mathbf{r}' \qquad \dot{\vec{r}} = \vec{\omega} \times \vec{r}$$

$$\dot{\vec{e}} = \mathring{\vec{e}} + \vec{\omega} \times \vec{e}$$

$$\vec{F}_{\text{fict}} = -m\,\vec{\omega} \times (\vec{\omega} \times \vec{r}) - 2m\,\vec{\omega} \times \vec{v} - m\,\dot{\vec{\omega}} \times \vec{r}$$

$$T_{\text{rot}} = \frac{1}{2}\sum_{\alpha,\beta} \omega_\alpha I_{\alpha\beta}\,\omega_\beta = \frac{1}{2}\sum_{\alpha,\beta} \omega'_\alpha I'_{\alpha\beta}\,\omega'_\beta = \frac{1}{2}(I_1\,\omega_1^2 + I_2\,\omega_2^2 + I_3\,\omega_3^2)$$

$$I_{\alpha\beta} = \sum_i m_i(r_i^2\delta_{\alpha\beta} - r_{i\alpha}r_{i\beta})$$

$$L_\alpha = \sum_\beta I_{\alpha\beta}\omega_\beta \qquad \vec{L} = I_1\omega_1\,\hat{1} + I_2\omega_2\,\hat{2} + I_3\omega_3\,\hat{3}$$

$$0 = \mathring{\vec{L}} + \vec{\omega} \times \vec{L}$$

$$\begin{aligned} I_1\dot{\omega}_1 &= (I_2 - I_3)\omega_2\omega_3 \\ I_2\dot{\omega}_2 &= (I_3 - I_1)\omega_3\omega_1 \\ I_3\dot{\omega}_3 &= (I_1 - I_2)\omega_1\omega_2 \end{aligned}$$

$$\Omega = (I_3/I - 1)\omega_3$$

$$\dot{\vec{\omega}} = \Omega\,\hat{3} \times \vec{\omega}$$

$$\dot{\vec{\omega}} = \omega_{\mathrm{p}}\,\hat{L} \times \vec{\omega} \qquad \dot{\hat{3}} = \omega_{\mathrm{p}}\,\hat{L} \times \hat{3}$$

$$\omega_{\mathrm{p}} = L/I \qquad \omega_{\mathrm{p}}\hat{L} = \vec{\omega} + \Omega\,\hat{3}$$

$$\cos\theta = \frac{I_3\,\omega_3}{I\,\omega_{\mathrm{p}}}$$