P3318 : LAST SECTION 26 APRIL

IN CLASS : $T = T_{trans} + T_{rot}$

$$T_{trans} = \frac{1}{2}MV^2$$

$$T_{rot} = \frac{1}{2}\sum_\alpha m_\alpha (\vec{\omega}\times\vec{r}_\alpha)^2 = \frac{1}{2}\sum_{ij} I_{ij}\omega_i\omega_j$$

$$I_{ij} = \sum_\alpha m_\alpha \left[\vec{r}_\alpha^{\,2}\delta_{ij} - (\vec{r}_\alpha)_i(\vec{r}_\alpha)_j \right]$$ ← COMPARE TO LEGENDRE DECOMPOSITION

continuous mass distribution

$$\sum_\alpha m_\alpha \to \int d^3\vec{r}\ \rho(\vec{r})$$

IN PARTICULAR :

$$\underline{I} = \int d^3\underline{r}\ \rho(\underline{r}) \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix}$$

can rotate s.t. $\underline{I} = \begin{pmatrix} I_1 & & \\ & I_2 & \\ & & I_3 \end{pmatrix}$

PRINCIPAL AXES PRINCIPAL MOMENTS

[examples from Tong, Marion & Thornton]

eg. the principal moments are $\mathbb{R}$, ≥ 0

pf. given an orbitrary vector $\vec{c}$

$$\vec{c}^T I \vec{c} = I_{ab} c^a c^b$$
$$= \sum_\alpha m_\alpha \left(\vec{r}_\alpha^{\,2} \vec{c}^{\,2} - (\vec{r}_\alpha \cdot \vec{c})^2\right)$$
$$\geq 0$$
$$\text{“=” only if all } \vec{r}_\alpha \text{ COLINEAR w/ } \vec{c}$$

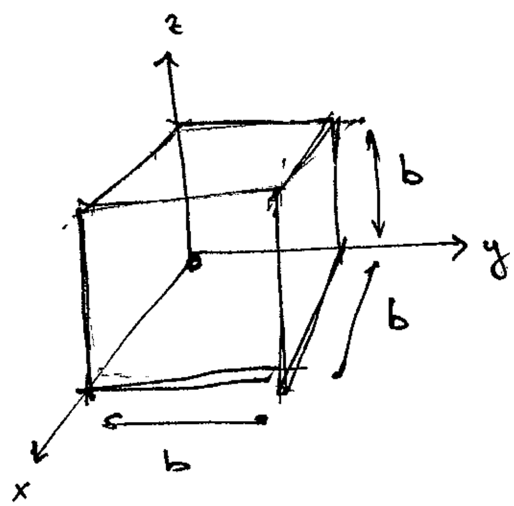
SUPPOSE $\vec{c}$ is the A^{th} EIGENVECTOR of I

THEN:

$$I_{ab} c_a c_b = I_A \vec{c}^{\,2}$$

$$\Rightarrow \boxed{I_A \geq 0}$$

eg.



HOMOGENEOUS; DENSITY ρ

TOTAL MASS M

ORIGIN NOT @ cm.

← silly choice, eh?

BRUTE FORCE THE FIRST COUPLE:

$$I_{11} = \rho \int_0^b dz \underbrace{\int_0^b dy(y^2+z^2)}_{\frac{1}{3}y^3 + yz^2 \big|_0^b} \underbrace{\int_0^b dx}_{b}$$

$$= \rho \int_0^b dz \left(\tfrac{1}{3}b^3 + bz^2\right) \cdot b$$

$$= \rho b \cdot \left(\tfrac{1}{3}b^4 + \tfrac{1}{3}b^4\right)$$

$$= \tfrac{2}{3}\rho b^5 \longleftarrow = \tfrac{2}{3}Mb^2$$

$$I_{12} = -\rho \int_0^b x\, dx \int_0^b y\, dy \int_0^b dz$$

$$= -\rho \left(\tfrac{1}{2}b^2\right) \cdot \left(\tfrac{1}{2}b^2\right) \cdot b$$

$$= -\tfrac{1}{4}\rho b^5 \longleftarrow -\tfrac{1}{4}Mb^2$$

phew. should we do the rest?

No: BY SYMMETRY, ALL DIAGONAL ELEM EQUAL & ALL OFF-DIAG ELEM EQUAL.

$$I_{11} = I_{22} = I_{33} \equiv \tfrac{2}{3}\beta \quad \checkmark$$

$$I_{12} = I_{13} = I_{23} \equiv -\tfrac{1}{4}\beta$$

obs: not diagonal... silly coordinates.

PRINCIPAL AXES

WANT: $I_{ij} = I_i \delta_{ij}$

then $L_i = \sum_j I_i \delta_{ij} \omega_j = I_i \omega_i$

$T_{rot} = \frac{1}{2} \sum_{ij} I_i \delta_{ij} \omega_i \omega_j = \frac{1}{2} \sum_i I_i \omega_i^2$

~~AXIS~~ BASIS WHERE ANGULAR VEL & ANGULAR MOMENTUM ARE ALONG THE SAME (PRINCIPAL) AXES.

$$\vec{L} = I\vec{\omega} \quad \leftrightarrow \quad \vec{L} - I\vec{\omega} = 0$$

$$\begin{pmatrix} I\omega_1 \\ \vdots \end{pmatrix} \qquad \begin{pmatrix} I_{11}\omega_1 + I_{12}\omega_2 + I_{13}\omega_3 \\ \vdots \end{pmatrix}$$

$$\begin{aligned} (I_{11} - I)\omega_1 + I_{12}\omega_2 + I_{13}\omega_3 &= 0 \\ I_{21}\omega_1 + (I_{22} - I)\omega_2 + I_{23}\omega_3 &= 0 \\ &\vdots \end{aligned}$$

Nontrivial solution:

$$\begin{vmatrix} (I_{11} - I) & I_{12} & I_{13} \\ I_{21} & (I_{22} - I) & I_{23} \\ I_{31} & I_{32} & (I_{33} - I) \end{vmatrix} = 0$$

characteristic polynomial.

Principal ~~AXES~~ moments of CUBE

$$\begin{vmatrix} \frac{2}{3}\beta - I & -\frac{1}{4}\beta & -\frac{1}{4}\beta \\ -\frac{1}{4}\beta & \frac{2}{3}\beta - I & -\frac{1}{4}\beta \\ -\frac{1}{4}\beta & -\frac{1}{4}\beta & \frac{2}{3}\beta - I \end{vmatrix} = 0$$

$$\begin{vmatrix} \left(-\frac{11}{12}\beta + I\right) & \left(\frac{11}{12}\beta - I\right) & 0 \end{vmatrix} = 0$$

$$\begin{vmatrix} -1 & 1 & 0 \end{vmatrix} \left(\frac{11}{12}\beta - I\right) = 0$$

Expand:

$$\left(\frac{11}{12}\beta - I\right)\left[\left(\frac{2}{3}\beta - I\right)^2 - \frac{1}{16}\beta^2 - \frac{1}{4}\beta\left(\frac{2}{3}\beta - I\right) - \frac{1}{16}\beta^2 \right]$$

[WORK] $\left(\frac{1}{6}\beta - I\right)\left(\frac{11}{12}\beta - I\right)\left(\frac{11}{12}\beta - I\right) = 0$

PRINCIPAL MOMENTS:

$$\boxed{\begin{aligned} I_1 &= \frac{1}{6}\beta \\ I_2 &= \frac{11}{12}\beta \\ I_3 &= \frac{11}{12}\beta \end{aligned}}$$

I_2, I_3: identical → I_1 is Axis of sym

can check PLUG IN $I = I_1 = \frac{1}{6}\beta$
INTO SYSTEM OF EQ. $\overleftrightarrow{I} - I\vec{\omega} = 0$

↳ OBTAIN: $\omega_1^{(1)} = \omega_2^{(1)} = \omega_3^{(1)}$

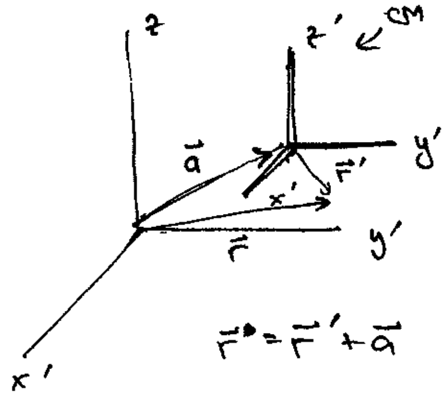
AXIS POINTS IN DIAGONAL.

other two are arb in plane PERP. TO THIS.

~~Rotation to the natural basis: transformation of a tensor:~~

~~$I \Rightarrow UIU^T$~~

Parallel axis thm



$\vec{r} = \vec{r}' + \vec{a}$

$$I_{ij} = \sum_\alpha m_\alpha \left(\delta_{ij} r_\alpha^2 - r_{\alpha i} r_{\alpha j} \right)$$

$$I_{ij} = \sum_\alpha m_\alpha \left(\delta_{ij} (\vec r\,'_\alpha + \vec a)^2 + (r'^i_\alpha + a^i)(r'^j_\alpha + a^j) \right)$$

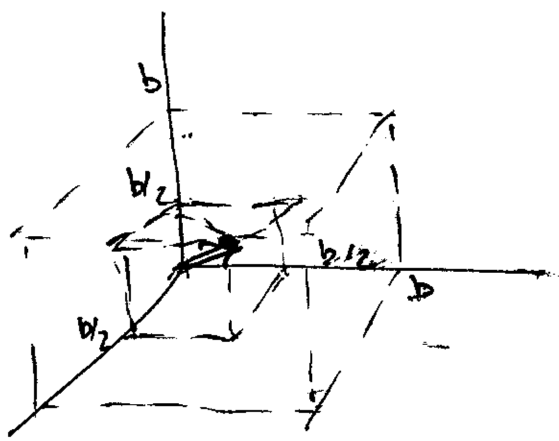
$$= I'_{ij} + \Big(\overset{M}{\sum_\alpha m_\alpha}\Big)\left(\delta_{ij} \vec a^{\,2} - a^i a^j\right)$$

$$+ \sum_\alpha m_\alpha \left(2 \delta_{ij} \vec r\,' \cdot \vec a - a^i r'^j_\alpha - r'^i_\alpha a^j\right)$$

last line vanishes by $\sum_\alpha m_\alpha r'^i = 0$
since we're in CM frame

$$\boxed{\underset{\text{not CM}}{I'_{ij}} = \underset{\text{CM}}{I_{ij}} - M\left(a^2 \delta_{ij} \;\; a_i a_j\right)}$$

e.g. cube w/ coord sys @ origin



PNS moment of inertia tensor is I'

$$I'_{11} = I'_{22} = I'_{33} = \tfrac{2}{3} M b^2$$
$$I'_{12} = I'_{23} = I'_{13} = -\tfrac{1}{4} M b^2$$

$$a_1 = a_2 = a_3 = b/2$$

then $I_{11} = I'_{11} - M(a^2 - a_1^2)$
$= I'_{11} - M(a_2^2 + a_3^2)$
$= \frac{2}{3}Mb^2 - 2\cdot(\frac{1}{4}Mb^2)$
$= \frac{1}{6}Mb^2$

$I_{12} = I'_{12} - M(-a_1 a_2)$
$= -\frac{1}{4}Mb^2 + (\frac{1}{4}Mb^2)$
$= 0$ ✓

others by symmetry.

$I = \mathrm{diag}(I_{11}, I_{11}, I_{11})$ ✓

$I \sim \mathbb{1}$

as long as origin is @ cm,
no ~~preffe~~ preferred axis

$I \to U I U^T = I$

CONTINUED FROM LAST TIME.

Dynamics of the falling cat

REPRESENTATIVE

$$\vec{L} = \sum m_\alpha \underbrace{\vec{r}_\alpha \times \dot{\vec{r}}_\alpha}_{\vec{\omega}_\alpha} \qquad \vec{r}_i = R\tilde{\vec{r}}_i$$

$$= \sum m_\alpha \Big[(R\tilde{r}_\alpha) \times (R\dot{\tilde{r}}_\alpha) + (R\tilde{r}_\alpha) \times (\dot{R}\tilde{r}_\alpha) \Big]$$

falling cat: $\vec{L} = 0$

[claim: $\Omega_{ab} = \varepsilon_{abc} \tilde{I}^{-1}_{ed} \tilde{L}_d$]

$$\tilde{I}_{ab} = \sum_\alpha m_\alpha \left[\tilde{r}_\alpha^2 \delta_{ab} - (\tilde{r}_\alpha)_a (\tilde{r}_\alpha)_b \right]$$

$$L_a = \varepsilon_{abc} \sum_\alpha m_\alpha \left[R_{bd} R_{ce} (\tilde{r}_\alpha)_d (\dot{\tilde{r}}_\alpha)_e + R_{bd} \dot{R}_{ce} (\tilde{r}_\alpha)_d (\tilde{r}_\alpha)_e \right]$$

mult by ε_{afg} use $\varepsilon_{abc}\varepsilon_{afg} = (\delta_{bf}\delta_{cg} - \delta_{bg}\delta_{cf})$

$$\varepsilon_{afg} L_a = \sum_\alpha m_\alpha \Big[R_{fd} R_{ge} (\tilde{r}_\alpha)_d (\dot{\tilde{r}})_e - (\text{DOT MOVE TO } \dot{R}_{ge}) - R_{gd} R_{fe} (\tilde{r}_\alpha)_d (\dot{\tilde{r}})_e + (\text{———} \ \dot{R}_{fe}) \Big]$$

multiply by $R_{fb} R_{gc}$

Use: $R_{fb} R_{fd} = (RR^T)_{bd} = \delta_{bd}$

$$R_{fb} R_{gc}\, \varepsilon_{afg} L_a = \sum_\alpha m_\alpha \Big[(\tilde{r}_\alpha)_b (\dot{\tilde{r}}_\alpha)_c - (\tilde{r}_\alpha)_c (\dot{\tilde{r}}_\alpha)_b - \Omega_{bd} (\tilde{r}_\alpha)_c (\tilde{r}_\alpha)_d + \Omega_{cd} (\tilde{r}_\alpha)_b (\tilde{r}_\alpha)_d \Big]$$

$$= 0$$

~~JUST LOOK @~~

LOOK @ APPARENT ANGULAR MOMENTUM

$$\tilde{L}_a = \varepsilon_{abc} \sum_\alpha m_\alpha (\tilde{r}_\alpha)_b (\dot{\tilde{r}}_\alpha)_c$$

$$= \cancel{R_{fb} R_{gc} \varepsilon_{afg} L_a}$$

$$=$$

$$\tilde{L}_1 = \sum_\alpha \Big[(\tilde{r}_\alpha)_2 (\dot{\tilde{r}}_\alpha)_3 - (\tilde{r}_\alpha)_3 (\dot{\tilde{r}}_2)_2 \Big]$$

$$= \sum_\alpha m_\alpha \Big[\Omega_{21} (\tilde{r}_\alpha)_3 (\tilde{r}_\alpha)_1 + \Omega_{23} (\tilde{r}_\alpha)_3 (\tilde{r}_\alpha)_3 - \Omega_{31} (\tilde{r}_\alpha)_2 (\tilde{r}_\alpha)_1 - \Omega_{32} (\tilde{r}_\alpha)_2 (\tilde{r}_\alpha)_2 \Big]$$

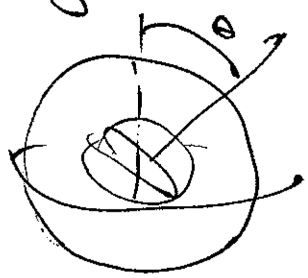
$$= \tilde{I}_{11} \Omega_{23} + \tilde{I}_{12} \Omega_{31} + \tilde{I}_{13} \Omega_{12}$$

$$\tilde{I}_{ab} = \sum_\alpha m_\alpha \left(\tilde{r}^2 \delta_{ab} - (\tilde{r}_\alpha)_a (\tilde{r}_\alpha)_b \right)$$

$$= \boxed{\tfrac{1}{2} \varepsilon_{abc} \tilde{I}_{1a} \Omega_{bc}}$$

A more direct example:

rotating concentric spheres



⇒ CAN CHANGE ^REL ORIENTATION w/o TOTAL ANGULAR MOMENTUM.

SEE: Gauge Kinematics of Deformable Bodies
Shapere & Wilczek