Lecture 7

- calculus of variations applied to mechanics : Hamilton's Principle

Recall that a necessary condition for the time functional T to be minimized by the function $\tilde{y}$ is that all the ~~funct~~ variational derivatives of T, at the "point" $\tilde{y}$, are zero :

$$\left.\frac{\delta T}{\delta y(x)}\right|_{\tilde{y}} = 0 \quad @ \text{ all } x$$

equivalently :

$$\left.\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial\left(\frac{dy}{dx}\right)}\right)\right|_{y=\tilde{y}} = 0 \qquad (1)$$

The structure of this equation is identical to the E-L equations of mechanics. Consider a system with one degree of freedom q. Now make the identifications,

$$y(x) \longleftrightarrow q(t)$$

$$\frac{dy}{dx} \longleftrightarrow \dot{q}$$

$$\frac{d}{dx} \longleftrightarrow \frac{d}{dt}$$

$$F \longleftrightarrow L$$

The solution of the E-L equation, let's call it $\tilde{q}$, now has another interpretation:

Given a system Lagrangian $L(q, \dot{q}, t)$, and the initial and final values of q in the time interval $[0, T]$, the trajectory (motion) taken by the system, $\tilde{q}(t)$, has the property that

$$S[q(t)] = \int_0^T L(q, \dot{q}, t)\, dt$$

is at an <u>extremum</u>.

Recall that our condition

$$\left.\frac{\delta S}{\delta q(t)}\right|_{\tilde{q}} = 0 \quad @ \text{ all } t$$

does not distinguish between "minimize" or "maximize", only that all the "slopes" are zero. In fact, we will see ~~[illegible]~~ that in a system as simple as the 1D harmonic oscillator the extremum is actually a saddle point, having both positive and negative curvature (in the multi-dimensional space).

The principle that a mechanical system takes the trajectory that extremizes the action S is called Hamilton's principle. Why a mechanical system would "want" something to be extremized is an intriguing question, whose answer had to await the discovery of quantum mechanics.