Lecture 5

- conjugate momentum
- non-holonomic constraints

The partial derivative

$$\frac{\partial L}{\partial \dot{q}_k} \equiv P_k$$

which shows up in the E.-L. equations and the definition of the Hamiltonian is called the "momentum conjugate to the generalized coordinate q_k".
When q_k is a cartesian position coordinate, or the angle of rotation

about a fixed axis, the conjugate momentum reduces to familiar forms of "momentum":

$V = 0$, $T = \frac{1}{2} m \dot{x}^2 = L$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} = P_x \quad \left(\begin{array}{l}\text{linear} \\ \text{momentum}\end{array}\right)$$

$$T = \frac{1}{2} I \dot{\theta}^2 = L$$

$$\frac{\partial L}{\partial \dot{\theta}} = I\dot{\theta} = P_\theta \quad \left(\begin{array}{l}\text{angular} \\ \text{momentum}\end{array}\right)$$

In general, P_k need not be a familiar type of momentum at all. Yet the property of it being conserved is preserved when a particular condition holds

②

conjugate momentum conservation:

If q_k is absent from L, then p_k is constant in time.

This follows directly from the E-L equation for q_k:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k} = 0 \quad \left(\begin{array}{l}\text{if } q_k \\ \text{is} \\ \text{absent}\end{array}\right)$$

$$\Rightarrow \frac{d}{dt} p_k = 0$$

The systems we have considered up to now, where, most generally the point-mass positions are expressible as

$$\vec{r}_i = \vec{r}_i(q_1, \ldots, q_N, t),$$

are said to have "holonomic constraints". To understand this term we need to examine a system to which this term does not apply.

Consider a wheel which is always upright on a flat surface upon which it rolls without slipping. How many degrees of freedom does this system have?

$\left(\begin{array}{c}\text{position of}\\\text{wheel center}\end{array}\right)$: x, y

$\left(\begin{array}{c}\text{angle of}\\\text{wheel axis}\end{array}\right)$: θ

$\left(\begin{array}{c}\text{angle of}\\\text{rotation about}\\\text{axis}\end{array}\right)$: ϕ

That's 4 DoF without taking into account the rolling-without-slipping constraints:

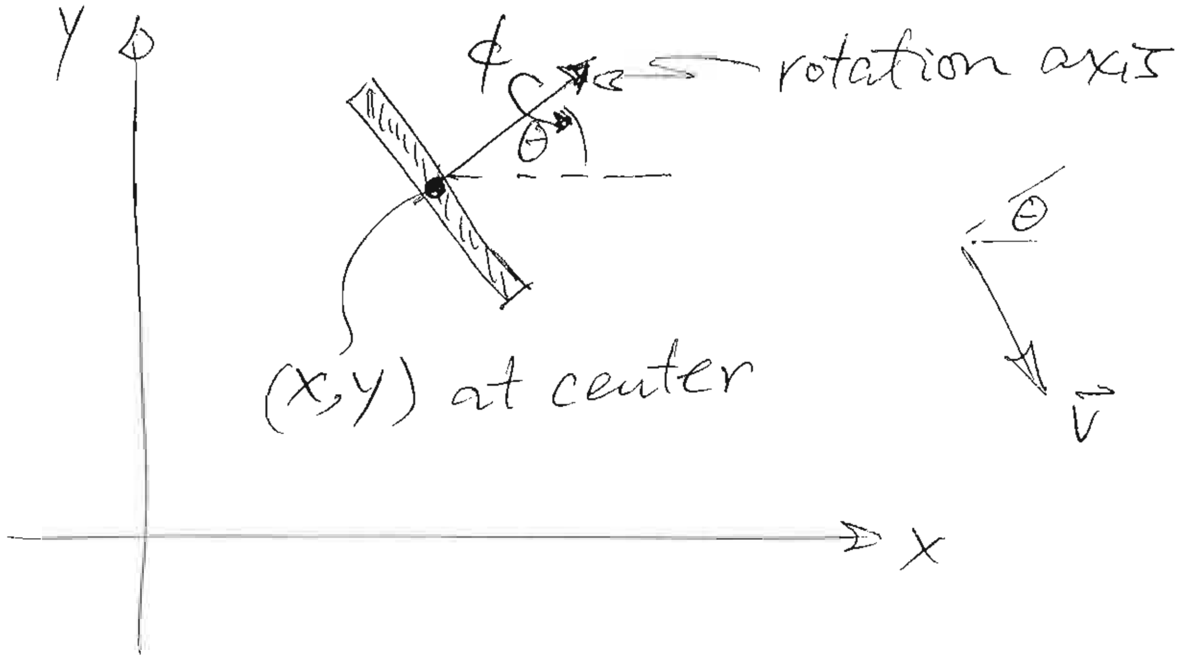
$$v_x^* = 0, \quad v_y^* = 0$$

These are the components of the

velocity of the point on the rim of the wheel that makes instantaneous contact with the surface.

Since each constraint subtracts one degree of freedom we have a system of $4-2=2$ DoF.

What should ~~I~~ we use for our two generalized coordinates? Previously we were told that this is mostly a question of convenience, since the two we choose should then determine all the others. So let's use θ and ϕ and see if we can then determine x and y from them.



$$\begin{pmatrix}\text{velocity of}\\ \text{center}\end{pmatrix} = \begin{pmatrix}\text{velocity of}\\ \text{contact point}\end{pmatrix} = r\dot{\phi} = v$$

~~(speed of contact point) = $r\dot{\phi} = v$~~

$$\dot{x} = v\sin\theta = r\dot{\phi}\sin\theta$$

$$\dot{y} = -v\cos\theta = -r\dot{\phi}\cos\theta$$

We would like to integrate these equations in time (remembering that $\theta(t)$ is in general time-dependent) and thereby arrive at formulas

$$x = x(\theta, \phi)$$
$$y = y(\theta, \phi).$$

But this is actually <u>impossible</u>.