Lecture 4

- equations of motion
- the Hamiltonian, H
- "conservation" of H
- H vs. E

The single scalar function $L = T - V$ contains all the information we need to obtain the motion of our system. The differential equations obtained in the previous are called the "equations of motion". There will be N equations of motion when our system has N degrees of freedom:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k}, \quad k=1,\ldots,N$$

"Euler-Lagrange equations"

We should remember that L can only be defined in the case of conservative forces, through the potential energy function V. Further, it is important to compute T and V in terms of coordinates in an inertial frame, since it is only in such a frame that Newton's 2nd Law is valid.

Let's obtain the equations of motion for the firefighter on the frictionless ladder. Recall the kinetic energy

$$T = \frac{1}{2} M \left(\frac{L^2}{4} \dot{\theta}^2 + L \cos\theta \, \dot{\theta} \, \dot{w} + \dot{w}^2 \right)$$

The only non-constraint force is ~~that~~ the gravitational force on the mass M with potential energy

$$V = Mgy = Mg \frac{L}{2} \cos\theta$$

We will consider the simple case where the wall is stationary, i.e. $\mathring{w} = 0$:

$$L = \frac{ML^2}{8}\dot{\theta}^2 - \frac{Mg L}{2}\cos\theta$$

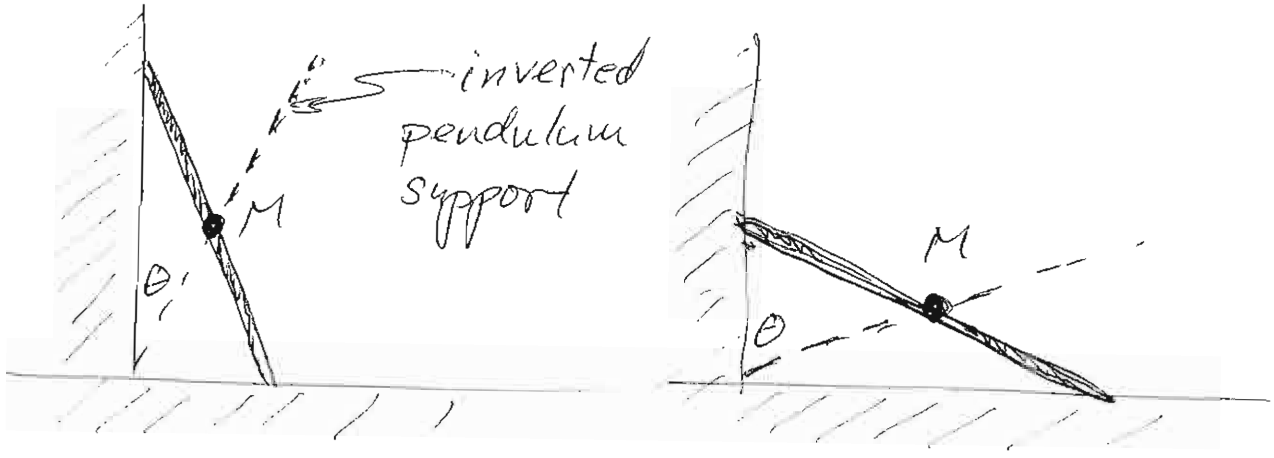
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{ML^2}{4}\ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = \frac{MgL}{2}\sin\theta$$

EOM : $$\frac{ML^2}{4}\ddot{\theta} = \frac{MgL}{2}\sin\theta$$

$$\ddot{\theta} = \frac{g}{L/2}\sin\theta$$

This is the equation of an inverted pendulum :



Q: To be more realistic, we should include the kinetic energy and weight of the ladder (up to now assumed massless). If the ladder has mass m and moment of inertia I (about its center), how does this change L?

A: $T \to T + \frac{mL^2}{8}\dot{\theta}^2 + \frac{1}{2} I \dot{\theta}^2$

$V \to V + \frac{mgL}{2}\cos\theta$

Starting with the function L we can define another function H, the "Hamiltonian", that has the nice property of being constant in time under very general conditions. Here is the definition:

$$H = \sum_k \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L$$

Here's what we get when we take the time derivative:

$$\frac{dH}{dt} = \sum_k \left(\ddot{q}_k \frac{\partial L}{\partial \dot{q}_k} + \dot{q}_k \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\right) - \frac{dL}{dt}$$

$$\frac{dL}{dt} = \sum_k \left(\frac{\partial L}{\partial q_k} \dot{q}_k + \frac{\partial L}{\partial \dot{q}_k} \ddot{q}_k \right) + \frac{\partial L}{\partial t}$$

Using the E-L eqn's in the first equation we obtain

$$\frac{dH}{dt} = \sum_k \left(\cancel{\ddot{q}_k \frac{\partial L}{\partial \dot{q}_k}} + \cancel{\dot{q}_k \frac{\partial L}{\partial q_k}} \right)$$

$$- \sum_k \left(\cancel{\frac{\partial L}{\partial q_k} \dot{q}_k} + \cancel{\frac{\partial L}{\partial \dot{q}_k} \ddot{q}_k} \right) - \frac{\partial L}{\partial t}$$

$$= - \frac{\partial L}{\partial t}$$

This tells us that $H =$ constant when L does not depend explicitly on time as it would as a result of a time-dependent environment.

When something is constant in time we say it is <u>conserved</u>.

In one of the homework problems you will show that when there is no time dependence in the "constraint equations",

$$\vec{r}_i = \vec{r}_i(q_1, \ldots, q_N)$$

then $H = T + V$, the total mechanical energy, E. So in this case we know that ~~both~~ H and E are equal and conserved.

It's possible for H to be different from E and yet still be conserved. Because of the homework exercise result we know we must have constraints of the form

$$\vec{r}_i = \vec{r}_i(q_1, \ldots, q_N, t)$$

However, we do not want t

to appear in L since then $\frac{\partial L}{\partial t} \neq 0$ and H will not be conserved. Here's a simple example:

1D particle in inertial moving frame:

$$x = \tilde{x} + vt$$

x = pos. in rest frame (r_x)

$\tilde{x}$ = gen. coordinate

$$\dot{x} = \dot{\tilde{x}} + v$$

$$T = \frac{1}{2} m\dot{x}^2 = \frac{1}{2} m(\dot{\tilde{x}}^2 + 2\dot{\tilde{x}}v + v^2) = L$$

$$\tilde{p} = \frac{\partial L}{\partial \dot{\tilde{x}}} = m\dot{\tilde{x}} + mv$$

$$H = \tilde{p}\dot{\tilde{x}} - L = \frac{1}{2} m\dot{\tilde{x}}^2 - \frac{1}{2} mv^2 \neq T = E$$

$\frac{\partial L}{\partial t} = 0 \Rightarrow$ H = const. but $H \neq E$

Is it possible for $H = E$, and yet H and E are not conserved? Again, we need time-dependence in the constraint equations except now we <u>want</u> t to appear in L so $\frac{\partial L}{\partial t} \neq 0$. Here's a simple example:

1D particle in time-dependent potential:

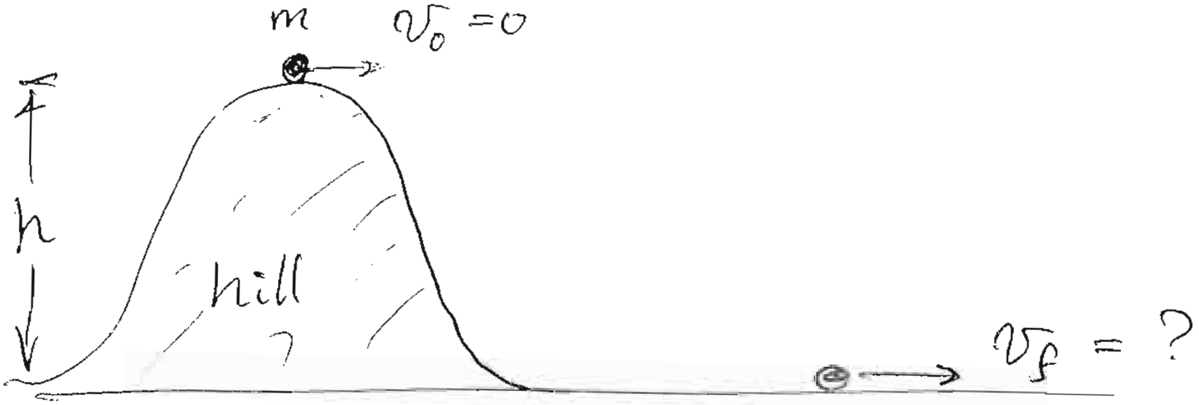
$$L = \frac{1}{2} m \dot{x}^2 - U(x,t)$$

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t} = \frac{\partial U}{\partial t} \neq 0$$

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$H = p\dot{x} - L = \frac{1}{2} m \dot{x}^2 + U(x,t) = E$$

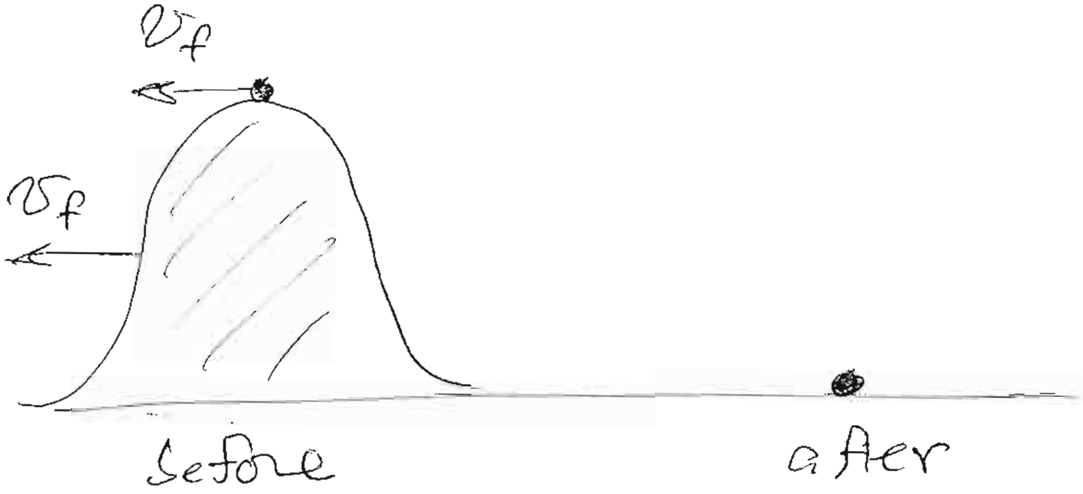
Let's revisit some freshman energy conservation problems.



When released from rest, we use energy conservation to find the final speed of the mass like this

$$mgh = \frac{1}{2} m v_f^2 \Rightarrow v_f = \sqrt{2gh}$$

~~Redo~~ Check this problem, again using energy conservation, but in a frame moving with the final velocity of the mass:



Q : Why is E not conserved ?