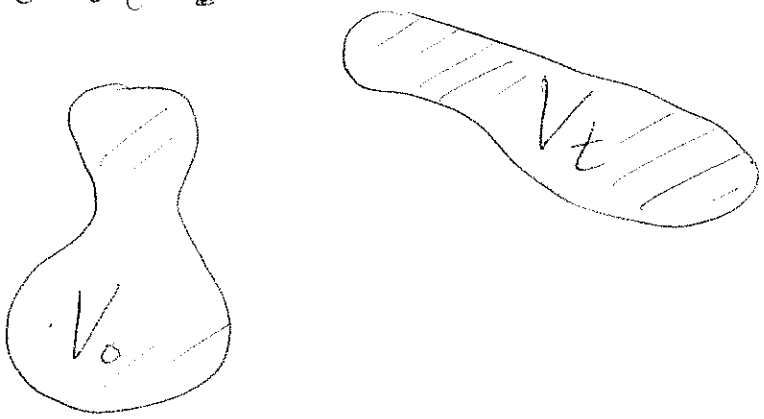
Dynamics in phase space

- Poincaré recurrence thm.
- action principle in phase space
- canonical transformations
- "Poisson brackets"

Consider a general dynamical system with some initial condition represented by some point A in phase space (e.g. A = all molecules in left half of box, moving upward). Let V_0 be an arbitrary neighborhood of A, representing initial conditions close to A. As we make V_0 smaller the set of initial conditions becomes more precisely defined.

Also, let V_t be the set (region) in phase space where all the initial conditions in V_0 flow to after time t. We'll assume there exists a t such that V_0 and V_t are disjoint:



If such a t did not exist, as might happen if V_0 is the neighborhood of a stable equilibrium point, then it would not be necessary to prove "recurrence". The statement of recurrence takes the following form:

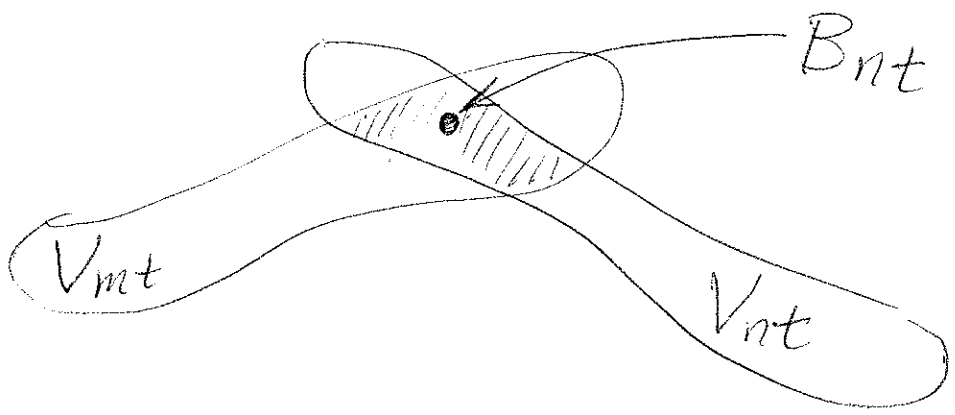
Poincare recurr. thm. : there exists a point $B_0 \in V_0$ and a finite time T such that B_0 flows to $B_T \in V_0$ in time T.

To prove this theorem we need to make use of the fact that the accessible phase space has finite total volume. Many dynamical systems have this property (molecules in a box) but certainly there are some that do not.

Q : Describe a system where the total accessible phase space volume is not finite.

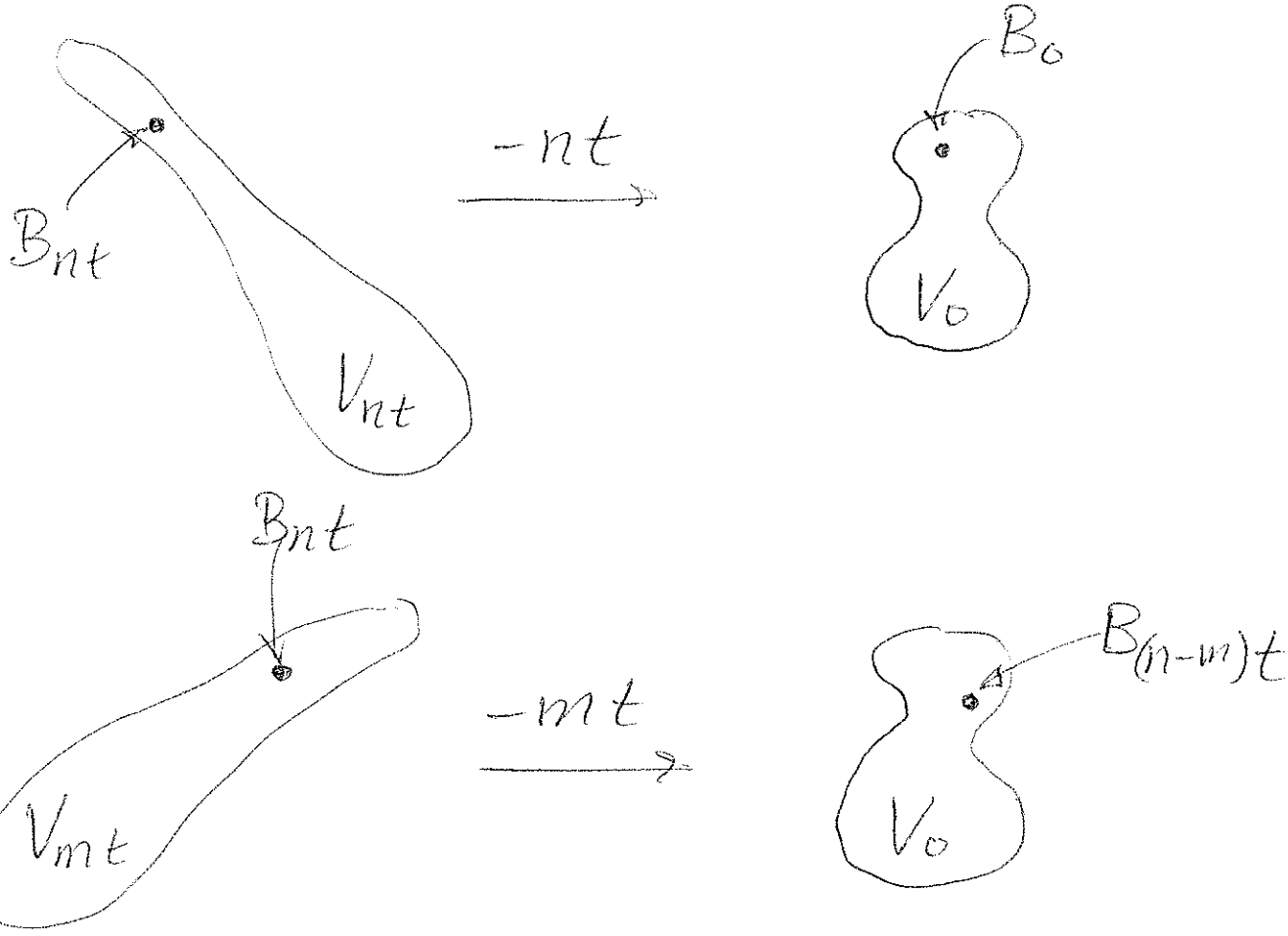
Proof : Consider the regions V_t, V_{2t}

V_{3t}, ... that V_0 flows to after times t, $2t$, $3t$, ... All of these regions have exactly the same volume by Liouville's thm. Moreover, this infinite sequence of regions cannot all be disjoint, since then the volume of accessible phase space would not be finite. Therefore, there must be regions V_{mt} and V_{nt} $(m<n)$ that intersect:



Let B_{nt} be a point in the intersection of these regions.

Now run time backwards: by $-nt$ on region V_{nt} and by $-mt$ on region V_{mt}:



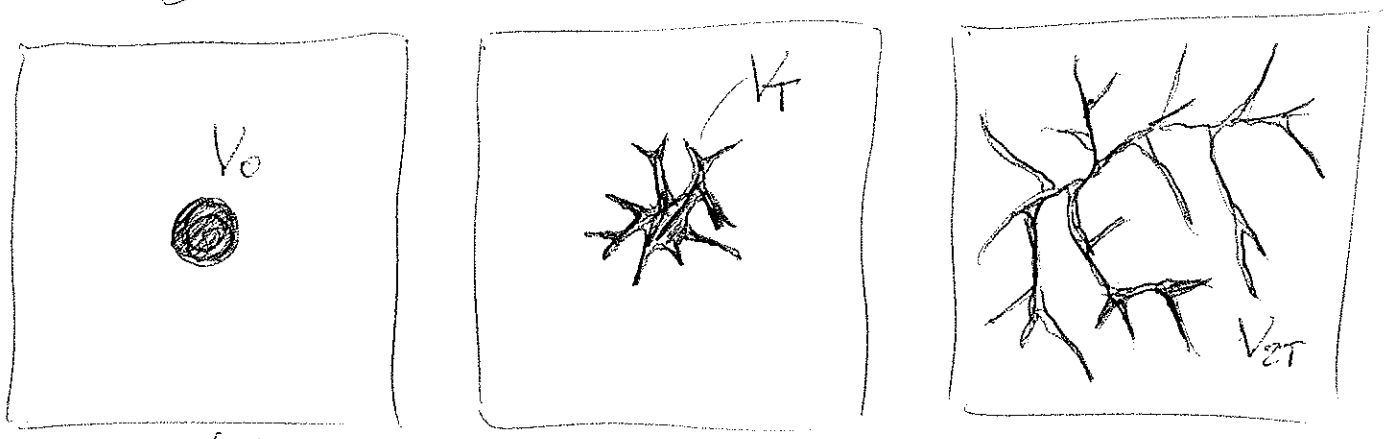
The drawings on the right show that $B_0 \in V_0$ flows to $B_T \in V_0$ after time $T = (n-m)t$. QED

The conclusion of the recurrence thm. seems at odds with what we were taught about the 2nd Law of Thermodynamics (i.e. how can all the molecules return to the left box-half without reducing the disorder/entropy ?).

Q: How do we resolve this? Does classical mechanics leave out some essential physics?

The resolution lies not in additional (or new) physics, but an appreciation for the high complexity of time-evolution in phase space. The phase space regions in our drawings have

been grossly misleading in their simplicity. A better rendition of the time evolution of V_0 is the following:

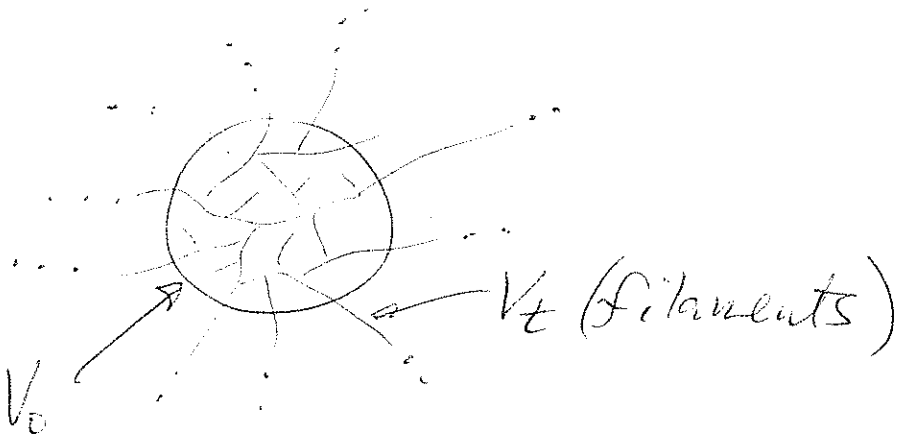


accessible phase space

$t=0$ $t=T$ $t=2T$

Even though V_T, V_{2T}, ... have the same (small) volume as V_0, V_t manages to ~~regularly part~~ spread out over all the accessible phase space as t increases (rapidly and also very uniformly in the case of a gas)

The fraction of V_t that lies within V_0 when t is large (such that V_t fills the accessible phase space uniformly) is very small, so that the subset of V_0 containing the "recurring" initial points B_0 is very small. It's true that the molecules will, after a long time, return to the left half of the box but only when their initial conditions are precisely fine tuned:



$$\text{fraction of recurring initial cond.} = \frac{\text{vol}(V_0 \cap V_t)}{\text{vol}(V_0)}$$

Hamilton's extremal action principle takes the following form in phase space. The trajectory now involves 2N functions of time $q_1(t), q_2(t), \ldots, p_1(t), p_2(t), \ldots$ and the action functional depends on all of them treated as independent functions:

$$S[q_1(t), \ldots, p_1(t), \ldots] = \int_{t_1}^{t_2} \left(\sum_i p_i \dot{q}_i - H \right) dt$$

We recognize the integrand as the Lagrangian. However now, to confirm the extremal action property, we need to independently

vary both q's and p's:

$$q_i(t) = \tilde{q}_i(t) + \delta q_i(t) \quad i=1,\dots,N$$

$$p_i(t) = \tilde{p}_i(t) + \delta p_i(t) \quad "$$

Here $\tilde{q}$ and $\tilde{p}$ are the functions that satisfy the equations of motion, i.e. Hamilton's equations.

Calculating the resulting variation of S we obtain:

$$\delta S = \int_{t_1}^{t_2} \Big(\sum_i (\delta p_i \dot{\tilde{q}}_i - \dot{\tilde{p}}_i \delta q_i) - \sum_i \Big(\frac{\partial H}{\partial q_i} \delta q_i + \frac{\partial H}{\partial p_i} \delta p_i \Big) \Big) dt + \underbrace{\sum_i \tilde{p}_i \delta q_i \Big|_{t_1}^{t_2}}_{0} + \begin{pmatrix} \text{second} \\ \text{order} \\ \text{terms} \end{pmatrix}$$

As in the Lagrangian formalism the variation of the q's is required to be zero at the endpoints of the trajectory (but there is no such requirement on δp's).

Combining terms,

$$\delta S = \int_{t_1}^{t_2} \sum_i \left(\delta p_i \left(\dot{q}_i - \frac{\partial H}{\partial p_i} \right) - \delta q_i \left(\dot{p}_i + \frac{\partial H}{\partial q_i} \right) \right) dt + \cdots$$

we see that δS vanishes to linear order in δp's and δq's because the coefficients of these variations ~~are Hamiltons~~ vanish by Hamilton's equations.

Turning things around (say we didn't know Hamilton's equations), Hamilton's principle in phase space is the statement that the true trajectory ($\tilde{q}$'s, $\tilde{p}$'s) has the property that the action S is extremal (zero first-order variation) for arbitrary and independent variations of the p's and q's.

In the Lagrangian formalism we often made use of the fact that we have enormous freedom in how a given dynamical system may be described by generalized coordinates

of our choosing. For example, we might have a 1DoF system with generalized coordinate q and Lagrangian $L(q, \dot{q}, t)$. But now suppose we find it more convenient to describe this system by a coordinate Q related to the old coordinate by

$$Q = Q(q, t).$$

Inverting this,

$$q = q(Q, t)$$

we can express the Lagrangian in terms of Q:

$$L(q, \dot{q}, t) = L\left(q(Q,t), \frac{\partial q}{\partial Q}\dot{Q} + \frac{\partial q}{\partial t}, t\right)$$

$$= L'(Q, \dot{Q}, t).$$

The original equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}$$

are then replaced by

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{Q}}\right) = \frac{\partial L'}{\partial Q} .$$

So the <u>form</u> of the equations of motion is unchanged (This follows immediately from the Lagrangian action principle, but can be checked by transforming the differential equation directly.)

We can try a similar coordinate + momentum transformation in the Hamiltonian

formalism (just 1 DoF for now):

$$Q = Q(q, P, t)$$
$$P = P(q, P, t)$$

However, in order to preserve the <u>form</u> of the Hamilton's equations of motion the functions $Q(\cdots)$ and $P(\cdots)$ are now not arbitrary, they must satisfy a particular property. Coordinate-momentum transformations satisfying this property are called "canonical".