# Dynamics in phase space

- Hamiltonian flow field
- zero-divergence of flow: Liouville's theorem
- quantum limits on resolving phase-space

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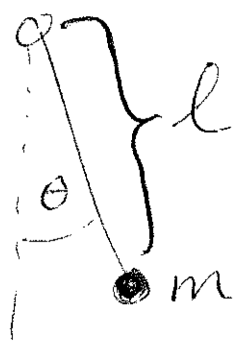
"Phase space" is the abstract space of which N dimensions correspond to the N generalized coordinates and another N to the conjugate momenta. With twice as many dimensions as ordinary coordinate space, phase space can convey, geometrically, not just one trajectory but the full range of motions that are possible. We will illustrate with the example of a pendulum that swings with arbitrary amplitude, even to the point of looping around its support.

$$L = \tfrac{1}{2} m l^2 \dot{\theta}^2 + m g l \cos\theta$$

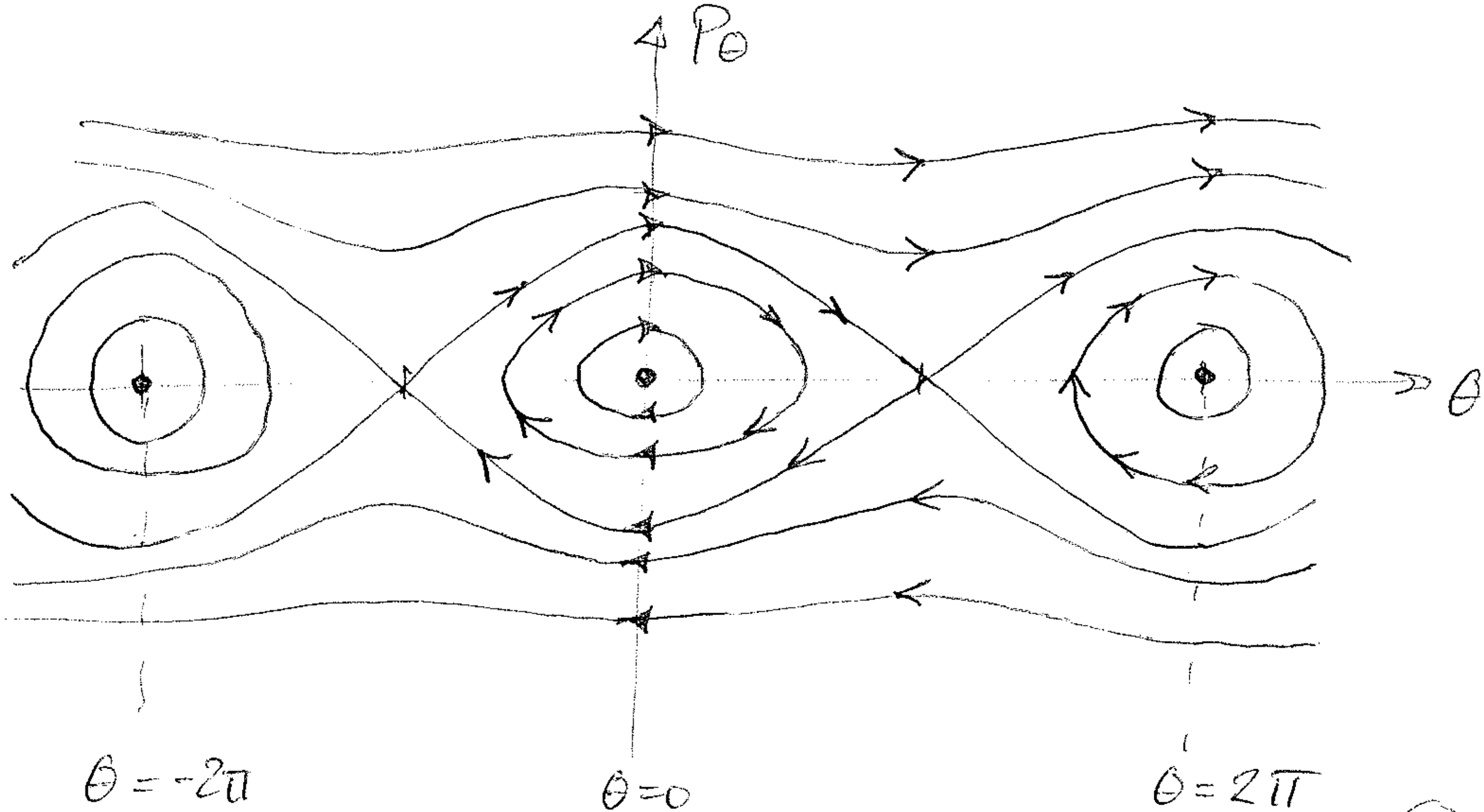



$l$ = fixed length of pend. arm, assumed massless

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m l^2 \dot{\theta}$$

$$H = p_\theta \dot{\theta} - L = \frac{p_\theta^2}{2 m l^2} - m g l \cos\theta$$

Since $\frac{dH}{dt} = -\frac{\partial L}{\partial t} = 0$, trajectories stay on the "surface" given by $H(p_\theta, \theta) = E =$ constant, where "surface" means one dimension less than the dimension of phase space, or $2N-1$. For our simple pendulum, the energy "surfaces" are one dimensional curves (contours):



To see how the system moves on the energy surfaces we recall Hamilton's equations:

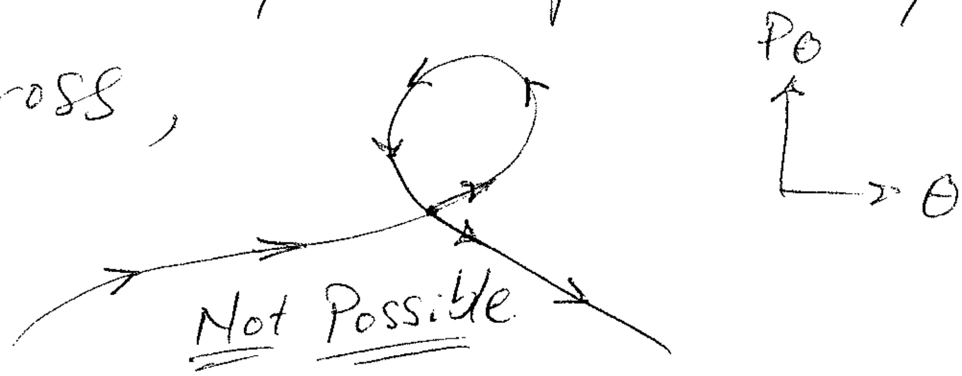
$$\dot{P}_\theta = -\frac{\partial H}{\partial \theta} = -mgl\sin\theta$$

$$\dot{\theta} = \frac{\partial H}{\partial P_\theta} = \frac{P_\theta}{ml^2}$$

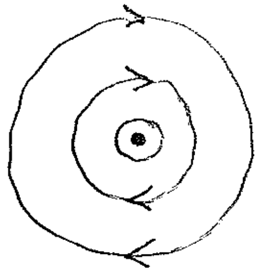
(See arrows added to drawing on page 3.) The "flow field" in phase space shows all possible motions:

- bounded oscillations
  $2\pi n - \theta_0 < \theta < 2\pi n + \theta_0 \qquad n = \text{integer}$
- clockwise spinning, $\dot{\theta} < 0$
- counterclockwise spinning, $\dot{\theta} > 0$

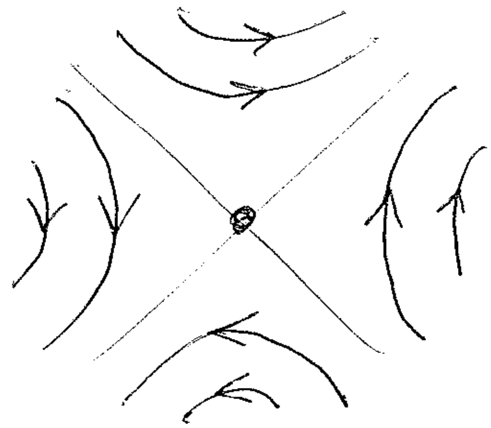
Trajectories in phase space may never cross,




since then there would not be a unique flow vector $(\dot{\theta}, \dot{p}_\theta)$ that depends only on the point in phase space $\left(\frac{\partial H}{\partial p_\theta}, -\frac{\partial H}{\partial \theta}\right)$. Points in phase space where the flow vector vanishes, $\dot{\theta} = \dot{p}_\theta = 0$, are <u>equilibrium points</u>: the trajectory stays at these points for all time. Our pendulum example has two types of equilibrium points:



stable equilibrium



unstable equilibrium

When the system is perturbed away from a stable equilibrium point, the motion will remain close to the equilibrium point. In contrast, when perturbed arbitrarily slightly from an unstable point, the motion will move away from the point (in an accelerated sense, as the magnitude of the flow vector increases away from the eq. point).

We now turn to a fundamental property of the flow field.

Liouville's Theorem:

The Hamiltonian flow field has zero divergence

We use $\vec{f}$ for the vector field of Hamiltonian flow, with 2N components

$$(\dot{q}_1, \dot{q}_2, \dots, \dot{q}_N, \dot{p}_1, \dot{p}_2, \dots, \dot{p}_N)$$

each of which is a function of the q's and p's by Hamilton's equation. Let's calculate the divergence of $\vec{f}$:

$$\vec{\nabla}\cdot\vec{f} = \frac{\partial \dot{q}_1}{\partial q_1} + \frac{\partial \dot{q}_2}{\partial q_2} + \cdots + \frac{\partial \dot{p}_1}{\partial p_1} + \frac{\partial \dot{p}_2}{\partial p_2} + \cdots$$

Next, use Hamilton's equations; for example

$$\frac{\partial \dot{q}_1}{\partial q_1} = \frac{\partial}{\partial q_1}\left(\frac{\partial H}{\partial p_1}\right) = \frac{\partial^2 H}{\partial q_1 \partial p_1}$$

$$\frac{\partial \dot{p}_1}{\partial p_1} = \frac{\partial}{\partial p_1}\left(-\frac{\partial H}{\partial q_1}\right) = -\frac{\partial^2 H}{\partial p_1 \partial q_1}$$

But since the order of the partials doesn't matter (for systems with well behaved $H$) we see immediately that

$$\vec{\nabla}\cdot\vec{f} = 0.$$

The significance of Liouville's thm. emerges when we consider not just a single trajectory. So consider very many trajectories, and suppose at time $t=0$ they are uniformly distributed points in phase space. The content of Liouville's ~~thm~~ theorem is equivalent to the statement that these points remain uniformly distributed for all time; for example, the following cannot happen: