# Kinematics in rotating frames

- fictitious forces
- Foucault pendulum

---

In the previous lecture we derived

$$\dot{\vec{e}} = \vec{\omega} \times \vec{e} + \overset{\circ}{\vec{e}}$$

where $\dot{\vec{e}}$ is the time derivative of a general vector and $\overset{\circ}{\vec{e}}$ is the time derivative expressed in the basis of body frame vectors and which ignores the fact these basis vectors are themselves in motion! Taking as an example $\vec{e} = \hat{x}$, a fixed vector in the body frame ($\overset{\circ}{\hat{x}} = 0$), we see

$$\overset{\circ}{\hat{x}} = \vec{\omega} \times \hat{x}$$

and similarly for $\hat{y}$ and $\hat{z}$. So the body axes are precessing about the instantaneous angular velocity vector $\vec{\omega}$. The usefulness of the definition $\overset{\circ}{\vec{e}}$ is that this is exactly how an observer ~~fixed~~ on the earth would define a rate of change on the assumption that axes fixed to the earth (north, south, etc.) are static with respect to an inertial frame (which they are not). The time derivative relationship thus provides a way to reconcile observations made in a non-inertial frame to the laws of physics that apply in inertial frames.

The first application of the time derivative transformation law is to $\vec{e} = \vec{\omega}$, which in general is time dependent:

$$\dot{\vec{\omega}} = \underbrace{\vec{\omega} \times \vec{\omega}}_{0} + \mathring{\vec{\omega}}$$

So the two forms of time derivative, in the case of $\vec{\omega}$, are equal.

We next apply the transformation to the velocity vector of a particle, $\vec{e} = \dot{\vec{r}} = \vec{\omega} \times \vec{r} + \mathring{\vec{r}}$.

$$\dot{\vec{e}} = \ddot{\vec{r}} = \vec{\omega} \times \dot{\vec{r}} + \dot{\mathring{\vec{r}}}$$

$$= \vec{\omega} \times (\vec{\omega} \times \vec{r} + \mathring{\vec{r}})$$

$$+ \mathring{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \mathring{\vec{r}} + \mathring{\mathring{\vec{r}}} .$$

$$\ddot{\vec{r}} = \mathring{\mathring{\vec{r}}} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + 2\vec{\omega} \times \mathring{\vec{r}}$$

$$+ \dot{\vec{\omega}} \times \vec{r} .$$

Now the true force acting on the particle is $m\ddot{\vec{r}}$, where $m$ is the particle mass. On the other hand, an observer in the body frame would interpret the acceleration in that frame in terms of an apparent force

$$\vec{F}_{fict} = m\overset{\circ\circ}{\vec{r}}.$$

The subscript stands for "fictitious", since some of the apparent acceleration is an artifact of the non-inertial frame. The true and fictitious forces are related in this way:

$$\vec{F}_{fict} = \vec{F} - m\vec{\omega}\times(\vec{\omega}\times\vec{r})$$
$$-2m\vec{\omega}\times\mathring{\vec{r}} - \dot{\vec{\omega}}\times\vec{r}$$

The $O(\omega^2)$ and velocity-independent term is called the "centrifugal" pseudo force. The $O(\omega)$ and velocity-dependent term is the "Coriolis" pseudo force. Finally, the $O(\dot{\omega})$ term is absent in typical situations, such as earth-bound observations, where $\vec{\omega}$ is approximately constant.

---

We will use the formula for $\vec{F}_{fict}$ to explain the rotation of the plane of oscillation of the Foucault pendulum.

---

Q: Which term, centrifugal or coriolis, is the dominant one?

---

A: cent. : coriolis $= \omega \cdot \Delta r : \Delta v$

$\Delta r \sim l$ = length of pend. string

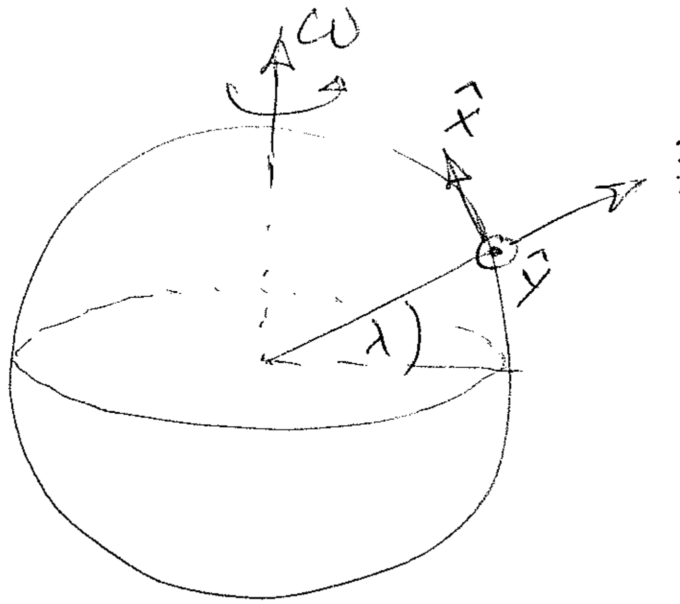
$\Delta v \sim \sqrt{gl}$ = pend. speed

$l \sim 1\,m$ : $\omega l \sim \frac{2\pi}{24 \times 3600}$ (m/s)

$\sqrt{gl} \sim 3$ m/s

$\Longrightarrow$ <u>Coriolis</u> force is dominant

The diagram below shows our choice of body axes appropriate for a pendulum fixed to the rotating earth at latitude $\lambda$ :




Q : Shown is the frame one would use for a pendulum in the ~~n~~ <u>northern</u> hemisphere. How do we know the direction of $\vec{\omega}$ is correct ?

By elementary geometry:

$$\vec{\omega} = \Omega(\cos\lambda\,\hat{x} + \sin\lambda\,\hat{z})$$

$$\Omega = \frac{2\pi}{24\text{ hours}}$$

Assuming small angle oscillations, we can neglect the $\hat{z}$ component of velocity in the coriolis term:

$$-2m\vec{\omega}\times\vec{v} \triangleq -2m\Omega\big((\cos\lambda)v_y\hat{z} + (\sin\lambda)v_x\hat{y} - (\sin\lambda)v_y\hat{x}\big)$$

We are interested mostly in the motion in the x-y plane (the tangent plane to the earth at our latitude). Henceforth we restrict our attention to Newton's equations of motion for $\ddot{x}$ and $\ddot{y}$. Gravity provides the "true" restoring forces

$$F_x = -m\omega_g^2 x$$
$$F_y = -m\omega_g^2 y$$

where $\omega_g = \sqrt{g/\ell}$. The fictitious forces acting in the body frame are then

$$F_{fx} = -m\omega_g^2 x + 2m\Omega \sin\lambda \, \dot{y}$$
$$F_{fy} = -m\omega_g^2 y - 2m\Omega \sin\lambda \, \dot{x} \, .$$

Here are the resulting equations of motion:

$$\ddot{x} = -\omega_g^2 x + 2\omega_p^* \dot{y}$$

$$\ddot{y} = -\omega_g^2 y - 2\omega_p^* \dot{x}$$

where $\omega_p^* = \Omega \sin\lambda$. To solve these two equations we can use the trick that these are the real and imaginary parts of the single equation for $z = x + iy$ given by

$$\ddot{z} = -\omega_g^2 z - i 2\omega_p \dot{z} .$$

Basic solutions of this (linear, homogeneous) ~~a this~~ equation

are given by $z = e^{i\alpha t}$
where $\alpha$ satisfies the equation

$$-\alpha^2 = -\omega_g^2 + 2\omega_p \alpha \quad .$$

Since $\omega_g \gg \omega_p$, the solution to $0^{th}$ order in $\omega_p$ is

$$\alpha = \pm \omega_g \, .$$

We substitute this $0^{th}$ order $\alpha$ into the small term to get the $1^{st}$ order solution:

$$-\alpha^2 = -\omega_g^2 \pm 2\omega_p \omega_g$$

$$\alpha^2 = \omega_g^2 (1 \mp 2 \omega_p/\omega_g)$$

$$\alpha \cong \pm \omega_g \left(1 \mp \omega_p/\omega_g\right)$$

The two approximate $\alpha$'s are therefore

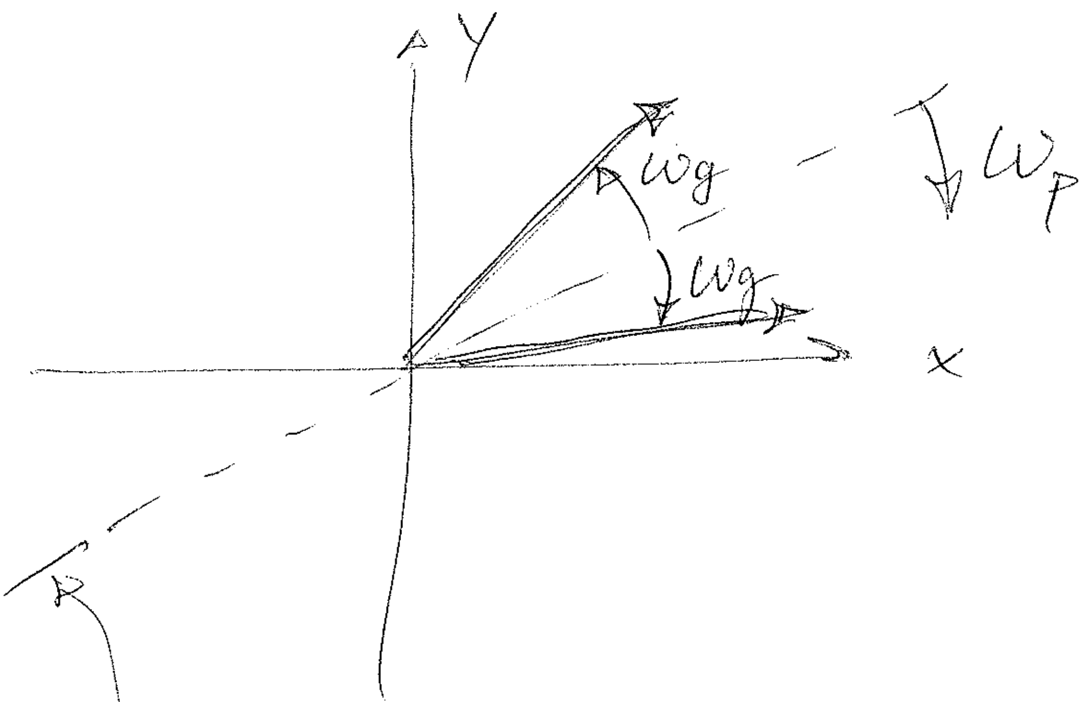
$$\alpha_+ = \omega_g - \omega_p$$

$$\alpha_- = -\omega_g - \omega_p$$

Combining the two basic solutions with equal amplitudes we get

$$z = A\left(e^{i(\omega_g - \omega_p)t} + e^{i(-\omega_g - \omega_p)t}\right)$$

which is the sum of two counter-rotating phasors, but each rotating relative to the <u>same</u> origin in phase, $e^{-i\omega_p t}$:




phasor-sum oscillates along this line, which rotates slowly clockwise with angular velocity

$$\omega_p = \Omega \sin \lambda$$