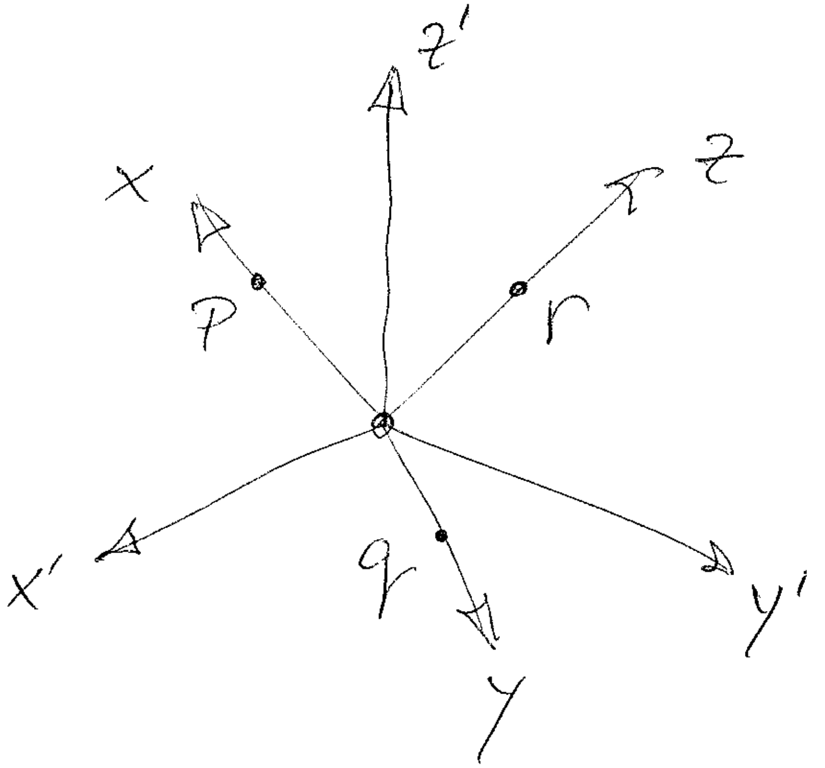
Kinematics in rotating frames

- "body" vs. "space" frame
- rotation matrices
- time-dependent rotations & instantaneous angular velocity
- transforming the time-derivative

Consider an inertial "space" frame K′ and another coordinate frame K that has been rotated with respect to K′. We will (eventually) think of K as fixed relative to some rigid body, so it is called the "body frame".



The points p, q, r have simple coordinates in the <u>body</u> frame:

$$p: \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad q: \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad r: \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

In the <u>space</u> frame we write their coordinates as:

$$p: \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}' \quad q: \begin{pmatrix} q_x \\ q_y \\ q_z \end{pmatrix}' \quad r: \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix}'$$

The primes remind us these coordinates are in the space frame, K'. By construction, since these points are at unit distance from the origin along coordinate axes (of the body frame), we know $p_x^2 + p_y^2 + p_z^2 = 1$ (etc. for q, r)

and $p_x q_x + p_y q_y + p_z q_z = 0$

(etc. for other pairs)

We can arrange these 9 numbers into a 3X3 matrix like this:

$$U = \begin{pmatrix} p_x & q_x & r_x \\ p_y & q_y & r_y \\ p_z & q_z & r_z \end{pmatrix}$$

The process of transforming coordinates from the body frame to the space frame is then accomplisled by matrix multiplication. For example:

$$U \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}' .$$

The properties we identified earlier, that the columns of U are orthonormal, define an orthogonal matrix. In

matrix notation the ortho-
normality of the columns is
expressed as

$$U^T U = \mathbb{1} = \text{identity matrix}$$

where

$$U^T = \begin{pmatrix} p_x & p_y & p_z \\ q_x & q_y & q_z \\ r_x & r_y & r_z \end{pmatrix}$$

is the transpose of U.

Q: What can we say about
the <u>rows</u> of U, are they
orthonormal too?

A: The relation $U^T U = \mathbb{1}$ also tells us that U^T is the matrix inverse of U, and since left and right matrix inverses are equal (for non-singular U), we know $UU^T = \mathbb{1}$. But this just tells us the rows are orthonormal.

Q: How many degrees of freedom does an orthogonal 3×3 matrix have?

A: There are 2 DoF for the first column, and then 1 DoF for the second (to keep it $\perp$ to the first). There are no continuous degrees of freedom left for the third column.

Notation: our shorthand for coordinate transformation from body to space frames will be

$$r' = Ur,$$

where from context we see that r' and r are column vectors (not scalar numbers).

Consider a point *fixed* in the body frame and allow for the rotation U to change with time. The coordinates in the space frame will then change and their velocity is given by

$$v' = \dot{r}' = \dot{U} r$$

$$= \dot{U}(U^T U) r$$

$$= \dot{U} U^T r'$$

we inserted the identity matrix $U^T U = \mathbb{1}$ so we could relate v' to r', i.e. the space-frame velocity in terms of the space-frame position. The combination

$$\dot{U} U^T = A'$$

defines an <u>anti</u>symmetric matrix.

Q: Why is A' antisymmetric?

A: Apply $\frac{d}{dt}$ to the identity

$$UU^T = \mathbb{1} \Rightarrow \dot{U}U^T + U\dot{U}^T = 0$$

Using the general matrix transpose property $(BC)^T = C^T B^T$, rewrite $U\dot{U}^T$ as $(\dot{U}U^T)^T$. Thus

$$\dot{U}U^T + (\dot{U}U^T)^T = 0$$

which tells us the matrix $\dot{U}U^T$ has zeros on the diagonal and the off-diagonals are paired with opposite signs.

An antisymmetric 3x3

matrix has 3 independent parameters, say ω_1', ω_2', ω_3' :

$$A' = \begin{pmatrix} 0 & -\omega_3' & \omega_2' \\ \omega_3' & 0 & -\omega_1' \\ -\omega_2' & \omega_1' & 0 \end{pmatrix}$$

With this parameterization we find

$$V' = A'r' = \begin{pmatrix} -\omega_3' y' + \omega_2' z' \\ \omega_3' x' - \omega_1' z' \\ -\omega_2' x' + \omega_1' y' \end{pmatrix}$$

which are the components of the cross product of $\begin{pmatrix} \omega_1' \\ \omega_2' \\ \omega_3' \end{pmatrix}$ with r'

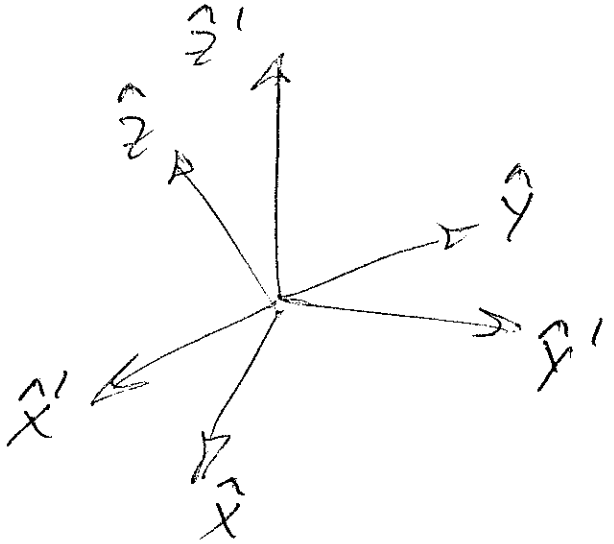
The equations we have written in terms of matrices and column vectors can also be written in terms of <u>vectors</u>. It's important to understand that <u>vectors</u> <u>are</u> <u>not</u> <u>the</u> <u>same</u> <u>thing</u> as <u>column vectors</u>. Consider the two column vectors for the position:

$$r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad r' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

Associated with the two frames (body & space) we also have two sets of <u>basis</u> <u>vectors</u>:

body basis : $\hat{x}, \hat{y}, \hat{z}$

space basis : $\hat{x}', \hat{y}', \hat{z}'$



The coordinates in each frame (column vectors), together with the corresponding basis, define the same frame-independent vector:

$$x\hat{x} + y\hat{y} + z\hat{z} = \vec{r} = x'\hat{x}' + y'\hat{y}' + z'\hat{z}'$$

As an example, the matrix/column vector relation

$v' = A'r'$ becomes $\vec{v} = \vec{\omega} \times \vec{r}$.

Let's extend our previous velocity calculation to allow for the possibility that the point in the body frame is a function of time. We'll also think of this "point" as a general vector $\vec{e}$ whose ~~co~~ <u>components</u> (previously: coordinates) are given by the column vector e in the body frame, and e' in the space frame:

$$\dot{e}' = \frac{d}{dt}(Ue) = \dot{U}e + U\dot{e}$$

$$= \dot{U}U^T Ue + U\dot{e}$$

$$= A'e' + (\dot{e})'$$

What vectors correspond to the terms in this equation?

As before

$$\dot{e}' \longrightarrow \dot{\vec{e}}$$

$$A'e' \longrightarrow \vec{\omega} \times \vec{e},$$

but what about $(\dot{e})'$? This term represents the velocity of $\vec{e}$

when the two frames are in a static relationship. We can also think of it as the velocity of $\vec{e}$ that ignores the fact the basis is also time-dependent.

We introduce the following notation for this kind of time derivative:

$$\overset{\circ}{\vec{e}} = \dot{e}_x \hat{x} + \dot{e}_y \hat{y} + \dot{e}_z \hat{z}$$

Contrast this with

$$\dot{\vec{e}} = \dot{e}'_x \hat{x}' + \dot{e}'_y \hat{y}' + \dot{e}'_z \hat{z}'$$

This is the true time derivative since the space basis $\hat{x}', \hat{y}', \hat{z}'$

is static. Our matrix/column-vector calculation shows the two forms of time derivative are related as follows:

$$\overset{\bullet}{\vec{e}} = \vec{\omega} \times \vec{e} + \overset{\circ}{\vec{e}}$$