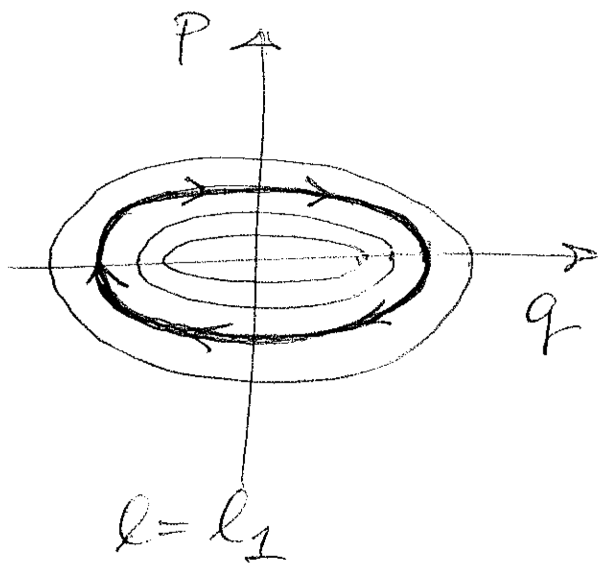
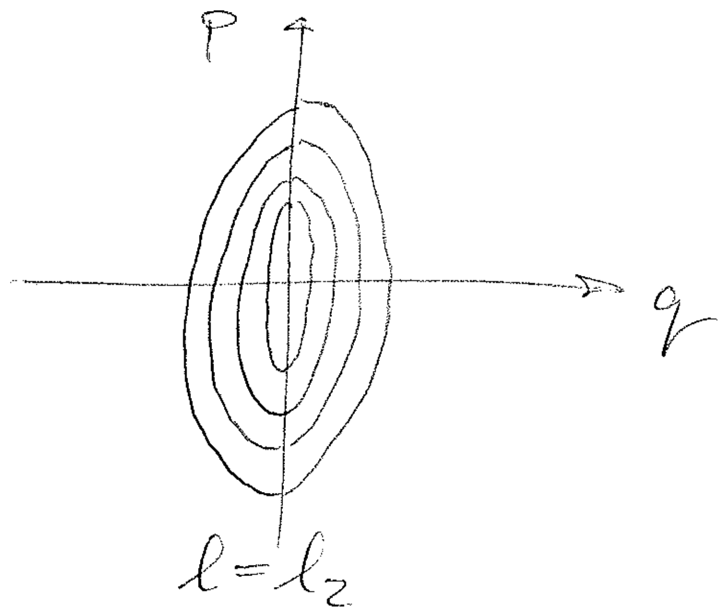
Adiabatic variation of the pendulum

Consider the l-dependence of our pendulum Hamiltonian:

$$H = \frac{p^2}{2ml^2} + \frac{1}{2} m g l q^2$$

We will be interested in contours of H in phase space for $l = l_1 =$ small and $l = l_2 =$ large:





Now, suppose at time $t=0$ we

have $\ell(0)=\ell_1$ and the pendulum orbit is the one shown by arrows on the left diagram. Over time we then slowly increase ℓ to $\ell(T)=\ell_2$, after which the pendulum will orbit on one of the $H=$ constant contours shown on the right diagram. But which one?

There is no reason for the pendulum to have the same energy at the two values of ℓ, since energy is not conserved when the Hamiltonian is time-dependent. However, something else is conserved in the limit of very slow change, the action. We will show this in a future lecture. For now, let's examine the consequences, assuming it is true.

Let q_0 be the maximum amplitude (for $p=0$) and p_0 the maximum momentum (for $q=0$). Then

$$\frac{p_0^2}{2ml^2} = \frac{1}{2} mgl\, q_0^2 \, .$$

The action for this motion is

$$I = \frac{1}{2\pi}(\text{phase-space area}) = \frac{1}{2\pi}\pi q_0 p_0 \, .$$

Keeping track of only the l-dependence,

$$I = \text{const.} \implies q_0 P_0 = \text{const.}$$

$$\implies q_0 \left(l^{3/2} q_0 \right) = \text{const.}$$

$$\implies q_0 \propto \frac{1}{l^{3/4}}$$

This shows how the amplitude decreases when the string is lengthened. Moreover:

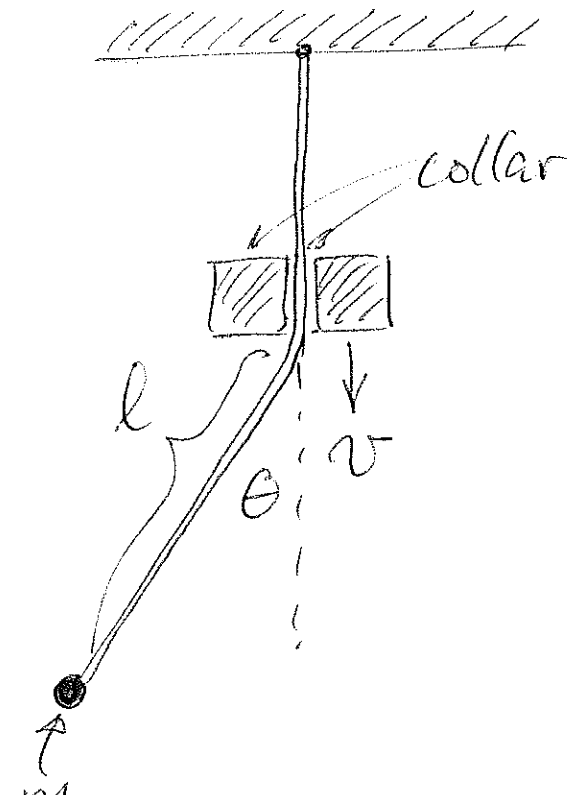
$$H \propto l q_0^2 \propto 1/l^{1/2}$$

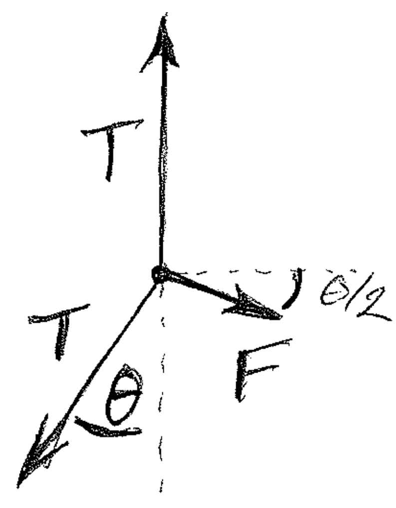
The dependence of the energy on l is more direct when we use action-angle variables:

$$H' = \omega I = \sqrt{g/l}\, I \propto 1/\sqrt{l}$$

since $\underline{I \text{ is constant}}$.

It's possible to check these scaling results for $q_0(\ell)$ $(=\theta_0(\ell))$ and $H(\ell)$ using elementary Newtonian mechanics. The idea is to evaluate the power input to the pendulum by the moving collar in the limit where the collar velocity v is small:





collar free-body diagram

T = string tension

F = force due to external agent

The first step is to determine the magnitudes of the forces T and F acting on the collar, and then the power provided by the external agent when moving the collar. Details have been scripted in the form of a homework problem.

Adiabatic invariance of the action

Our Newtonian analysis of the pendulum showed that, when the string length l was changed very slowly, the amplitude (and consequently the energy) could be explained from the apparent principle that the action is approximately constant in this limit. We would like to know the extent to which this "action-invariance" is true, since it is different from the symmetry-based conservation laws that are rigorously true.

Our method is to have the string length l depend on a dimensionless parameter s that varies between 0 and 1, where

$$l(0) = l_1 \quad , \quad l(1) = l_2 \, ,$$

and then set $s = \epsilon t$, where ϵ is a small parameter (with units of $time^{-1}$ or rate). As $\epsilon \to 0$ the process of changing the length becomes more adiabatic, requiring a total time $1/\epsilon$. By using the machinery of canonical transformations and generating functions we will ~~so~~ show the action is invariant up to terms of order ϵ^n, where

n may be arbitrarily large!
This fact is usually stated as "the action is invariant to all orders in the perturbation" ("perturbation" being non-constancy of the string length).

Recall the pendulum Hamiltonian:

$$H = \frac{p^2}{2ml^2} + \frac{1}{2} m \omega^2 l^2 q^2$$

This Hamiltonian has direct time dependence through l and also

$$\omega^2 = g/l .$$

In a previous lecture we transformed to action-angle variables using the generating function

$$F(q,\theta,t) = \tfrac{1}{2} m \sqrt{g}\, l^{3/2}(\epsilon t)\, q^2 \cot\theta$$

which is now also time-dependent. We have been careful to make the time dependence explicit by substituting $\sqrt{g/l}$ for ω. As we learned in the general discussion of generating functions, when these are time dependent the transformed Hamiltonian includes a term $\frac{\partial F}{\partial t}$:

$$H'(\theta,I,t) = I\omega +$$

$$\tfrac{3}{4} m\sqrt{g}\sqrt{l}\, \epsilon\, l'\, q^2 \cot\theta$$

$$\left(l' = \frac{dl}{ds}\right)$$

We remember to express H' only in terms of the new variables, θ and I :

$$q^2 \cot\theta = \frac{2I}{m\omega l^2} \sin^2\theta \cos\theta$$

$$H'(\theta, I, t) = I\omega + \frac{3}{2} I \sin\theta\cos\theta \left(\epsilon \frac{l'}{l}\right)$$

As a result of the new term, we see that I is no longer time-independent :

$$\dot{I} = -\frac{\partial H'}{\partial\theta} = -\frac{3}{2} I \cos 2\theta \left(\epsilon \frac{l'}{l}\right)$$

It appears the invariance of the action (I) is good only to order ϵ. However, we will show that the invariance is much better than that. The idea is to perform a sequence of canonical transformations, beginning with θ, I, which we rename θ_0, I_0:

$$H' = H_0(\theta_0, I_0, t) \xrightarrow{F_1} H_1(\theta_1, I_1, t) \xrightarrow{F_2} \text{etc.}$$

We also write H_0 in a more general way, so it applies not just to the pendulum:

$$H_0(\theta_0, I_0, t) = I_0\big(\omega + \epsilon h_0(\theta_0, \epsilon t)\big)$$

The generating function F_1 for the next pair of variables will be of the type where the new variable is the momentum, rather than the coordinate:

$$F_1(\theta_0, I_1, t) = I_1\left(\theta_0 - \frac{\epsilon}{\omega}\int_0^{\theta_0} h_0(\theta', \epsilon t)\,d\theta'\right)$$

Here are the two equations that relate the new variables to the old:

$$\left.\begin{aligned} I_0 &= \frac{\partial F_1}{\partial \theta_0} = I_1\left(1 - \frac{\epsilon}{\omega} h_0(\theta_0, \epsilon t)\right) \\ \theta_1 &= \frac{\partial F_1}{\partial I_1} = \theta_0 - \frac{\epsilon}{\omega}\int_0^{\theta_0} h_0(\theta', \epsilon t)\,d\theta' \end{aligned}\right\} *$$

Transforming the Hamiltonian involves substituting the new variables into the old Hamiltonian and adding the term $\frac{\partial F_1}{\partial t}$:

$$H_1 = H_0 + \frac{\partial F_1}{\partial t} = \text{(cont.)}$$

$$
\left.
\begin{aligned}
H_1 &= \underbrace{I_1 \left(1 - \tfrac{\epsilon}{\omega} h_0\right)}_{I_0}(\omega + \epsilon h_0) \\
&\quad - I_1 \frac{\epsilon}{\omega} \cdot \epsilon \int_0^{\theta_0} \frac{\partial h_0}{\partial s}(\theta', \epsilon t)\, d\theta' \\
&\quad + I_1 \frac{\epsilon}{\omega^2} \frac{\partial \omega}{\partial s} \epsilon \int_0^{\theta_0} h_0(\theta', \epsilon t)\, d\theta' \\
&= I_1 \left(\omega + \epsilon^2 h_1(\theta_1, \epsilon t)\right)
\end{aligned}
\right\} \quad (\#)
$$

where h_1 is obtained from h_0 and $\frac{\partial h_0}{\partial s}$, either evaluated at θ_0 or integrated over θ from 0 to θ_0. Here are the main facts that bear on our proof of invariance:

- The <u>form</u> of H_1 is the same as H_0, only the power of ϵ multiplying the perturbation is increased by 1.

- All time dependence, as in H_0, is through the combination $s = \epsilon t$.
- Repeating the procedure n times we obtain:

$$H_n = I_n(\omega + \epsilon^{n+1} h_n(\theta_n, \epsilon t))$$

The Hamiltonian H_n suggest we are practically done, since the equation of motion for I_n we get is

$$\dot{I}_n = -\frac{\partial H_n}{\partial \theta_n} = -\epsilon^{n+1} \frac{\partial h_n}{\partial \theta_n},$$

so that

$$I_n(T) = I_n(0) + O(\epsilon^{n+1}). \quad (\square)$$

But what we really wanted to

show was invariance of the original action variable, I_0 (not I_n). To argue this, we need to go back to the equations (*) which relate θ_0, I_0 to θ_1, I_1. Notice that if the function $h_0(\theta', s)$ vanishes at the endpoints $s=0$ and $s=1$ then

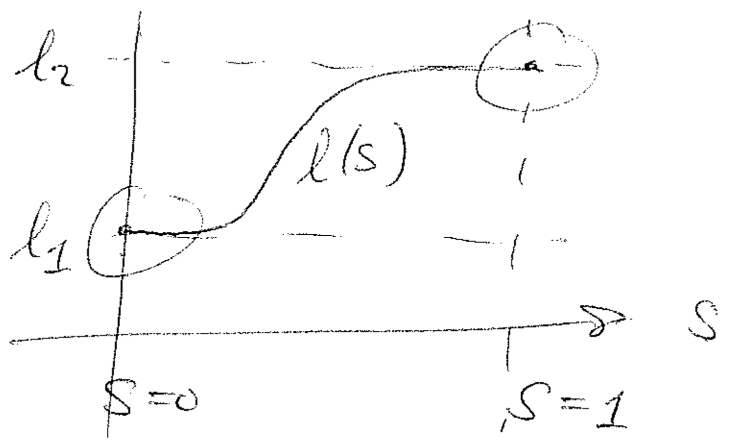
$$s=0: \quad I_0(0) = I_1(0), \quad \theta_0(0) = \theta_1(0)$$

$$s=1: \quad I_0(T) = I_1(T), \quad \theta_0(T) = \theta_1(T)$$

where T is defined as the time where $\epsilon T = s = 1$. If we wish to have <u>all</u> the pairs of variables θ_n, I_n equal θ_0, I_0 at the endpoints of the adiabatic process ($s=0$, $s=1$), then the property of $h_0(\theta', s)$ vanishing at the endpoints must extend to $h_n(\theta', s)$. Looking at the construction of h_1 (equations #) we see that h_1 will also have the desired property provided $\frac{\partial h_0}{\partial s}(\theta', s)$ also vanish at $s=0$ and $s=1$. Proceding to h_2, h_3, etc. we see that we want

$$0 = \frac{\partial^m h_0}{\partial s^m}(\theta', 0) = \frac{\partial^m h_0}{\partial s^m}(\theta', 1) \quad (*)$$

for all m. That is, we want the original perturbation function to have all terms in its Taylor series to be zero at $s=0$ and $s=1$. In the pendulum problem, h_0 depended on s as $h_0 \propto \frac{l'(s)}{l(s)}$, so our condition on h_0 translates to a very smooth start and stop in the variation of the string length:



Our condition will be satisfied if all derivatives of $l(s)$ vanish at the endpoints; there is no condition on $l(s)$ between the endpoints.

An example of a function, all of whose derivatives vanish at $s=0$ (and yet is not just a constant), is

$$l(s) = l_0 + A e^{-B/s^2} .$$

We can now conclude our proof. From the "smooth-start/stop" property (*) we have

$$I_0(0) = I_n(0), \quad \text{all } n$$

$$I_0(T) = I_n(T), \quad \text{all } n .$$

Using the high order invariance of $I_n(\square)$,

$$I_0(0) = I_n(0) = I_n(T) + O(\epsilon^{n+1})$$
$$= I_0(T) + O(\epsilon^{n+1})$$

QED.