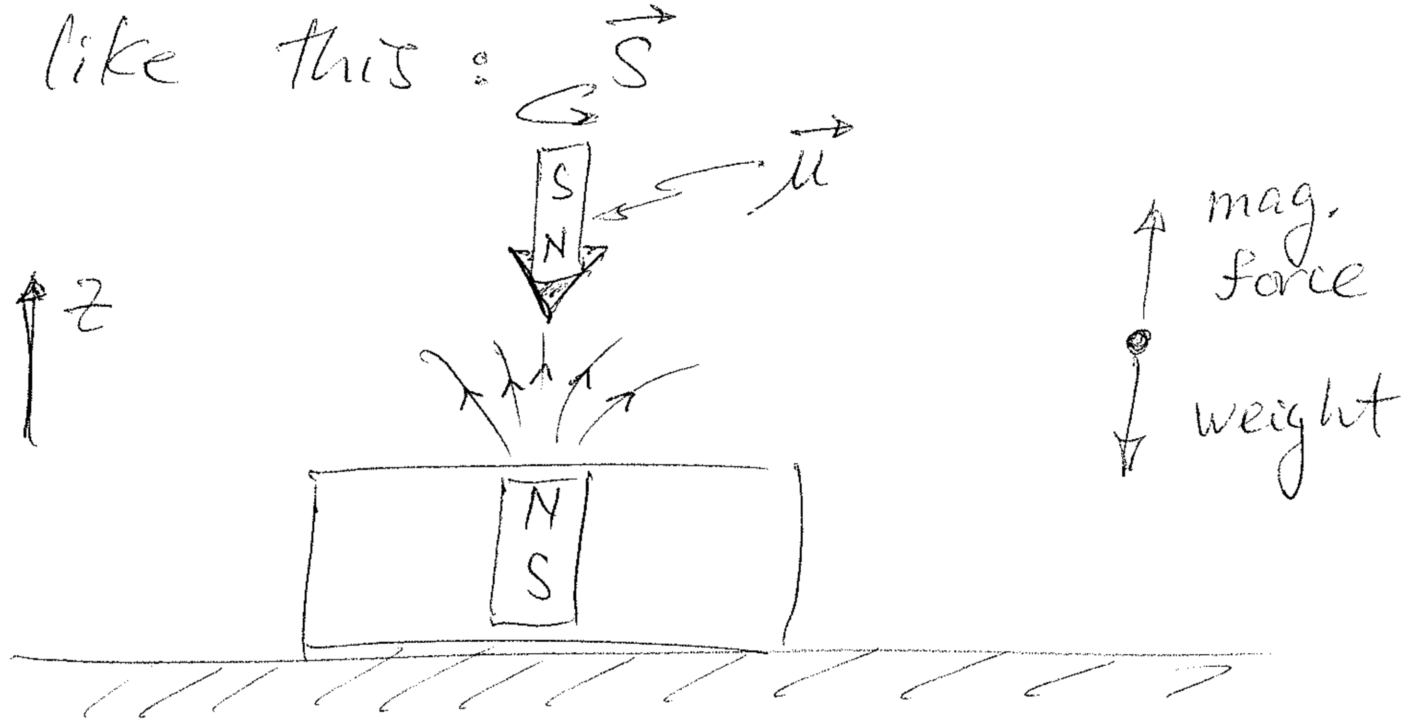
Applications of adiabatic invariance

- Levitron
- spin-1/2 particle in magnetic trap

The parameter, under whose variation some action variable I is adiabatically invariant, may actually be a dynamical (i.e. not external) variable of a larger system. A nice example of this is the levitating magnetic top, "Levitron".

Suppose we rapidly spin a top, whose axis is magnetized, in the field of a fixed magnet like this:



Suppose the spin $\vec{S}$ of the top is so large it keeps a fixed orientation in space (the torque required to change the axis would be too great). Let the fixed axis of the top be the z-axis, the direction of the gravitational ~~field~~ force. In this "fixed axis limit", the potential energy of top depends only on its position:

$$V(\vec{r}) = mgz - \vec{\mu}\cdot\vec{B}(\vec{r})$$

$$= mgz - \mu B_z(\vec{r})$$

This form of potential energy cannot explain the <u>stability</u> of the Levitron (Earnshaw's thm.). To see this, we express $\vec{B}$ as the gradient of a magneto-static potential Φ_B which (outside the source) satisfies the Laplace equation:

$$\vec{B} = \vec{\nabla}\Phi_B \ , \quad \nabla^2\Phi_B = 0$$

From these we obtain

$$B_z = \frac{\partial \Phi_B}{\partial z} \ , \quad \nabla^2 B_z = \frac{\partial}{\partial z}(\nabla^2\Phi_B) = 0.$$

And since the gravitational potential energy also satisfies the Laplace equation,

$$\nabla^2 V = \nabla^2(mgz) - \mu \nabla^2 B_z$$

$$= \quad 0 + 0 = 0.$$

The curvature of V, at a point of equilibrium, therefore cannot be positive (stable) in all three dimensions — at least one of the dimensions must be unstable.

If we carefully observe the motion of the top we notice there are actually <u>three</u> distinct time scales:

<u>fast</u> : spinning of top about its axis

<u>intermediate</u> : precession of spin axis

<u>slow</u> : center-of-mass motion

The second of these, precession, is key to the top's stability. In a constant magnetic field the precessional motion of the spin $\vec{S}$ is described the equation

$$\dot{\vec{S}} = \text{torque} = \vec{\mu} \times \vec{B}$$

where $\vec{\mu}$ is parallel to $\vec{S}$ when the spin is very fast (compared with precession). Therefore $\dot{\vec{S}}$ is always perpendicular to $\vec{S}$, thus

Keeping the magnitude of $\vec{S}$ constant at some value S. We can rewrite the precession equation as

$$\dot{\vec{S}} = \left(\frac{\mu B}{S}\right) \vec{S} \times \hat{b}$$

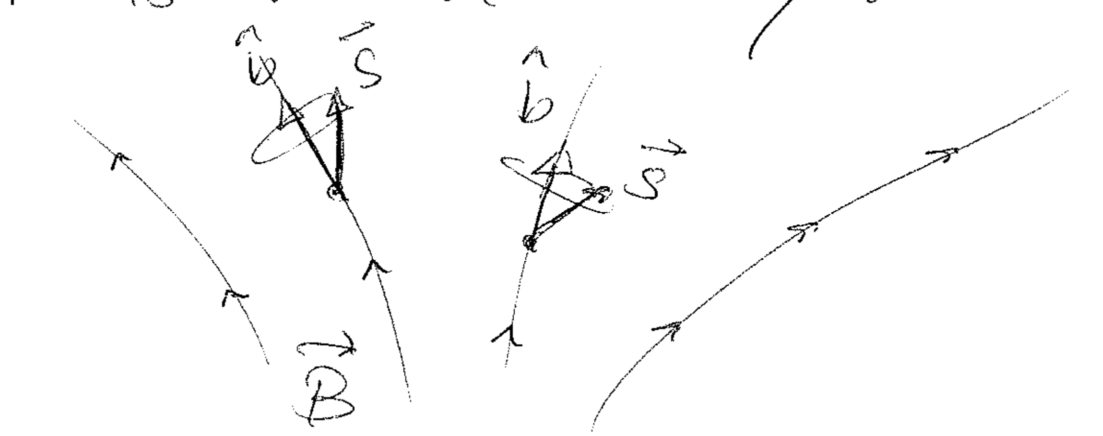
where $\hat{b}$ is a unit vector giving the direction of the magnetic field, and

$$\omega = \frac{\mu B}{S}$$

is the precession frequency. When ω is large, the spin axis precesses many times about the magnetic field axis $\hat{b}$, even when $\hat{b} = \hat{b}(\vec{r})$ is changing (slowly) as the top moves in a non-uniform $\vec{B}$. It seems reasonable, then, that $\vec{S}$ maintains a constant angle (or projection) with respect to $\hat{b}$. In fact, with a more formal analysis one can define the action variable

$$I = \vec{S} \cdot \hat{b}(\vec{r})$$

and use the fact that it is invariant when the "parameter" $\vec{r}$ is varied slowly:



Let Ω be the characteristic frequency of center-of-mass motion, the inverse of the time scale, T^{-1}, that describes the parameter change. On the other hand, ω is the frequency associated with the action variable I. Our previous study of adiabatic invariance showed that I is constant ~~by~~ to all orders (powers) in $\epsilon \propto \frac{\Omega}{\omega}$. What this really means is that the deviations ~~in~~ from constant I are exponentially small in $1/\epsilon \propto \omega/\Omega$; this will be negligible if the two frequency differ by as little as a factor of 20.

Let's use the invariance of

I to re-express the magnetic dipole energy:

$$-\vec{\mu}\cdot\vec{B}(\vec{r}) = -\left(\frac{\mu}{S}\vec{S}\right)\cdot\hat{b}(\vec{r})\,B(\vec{r})$$

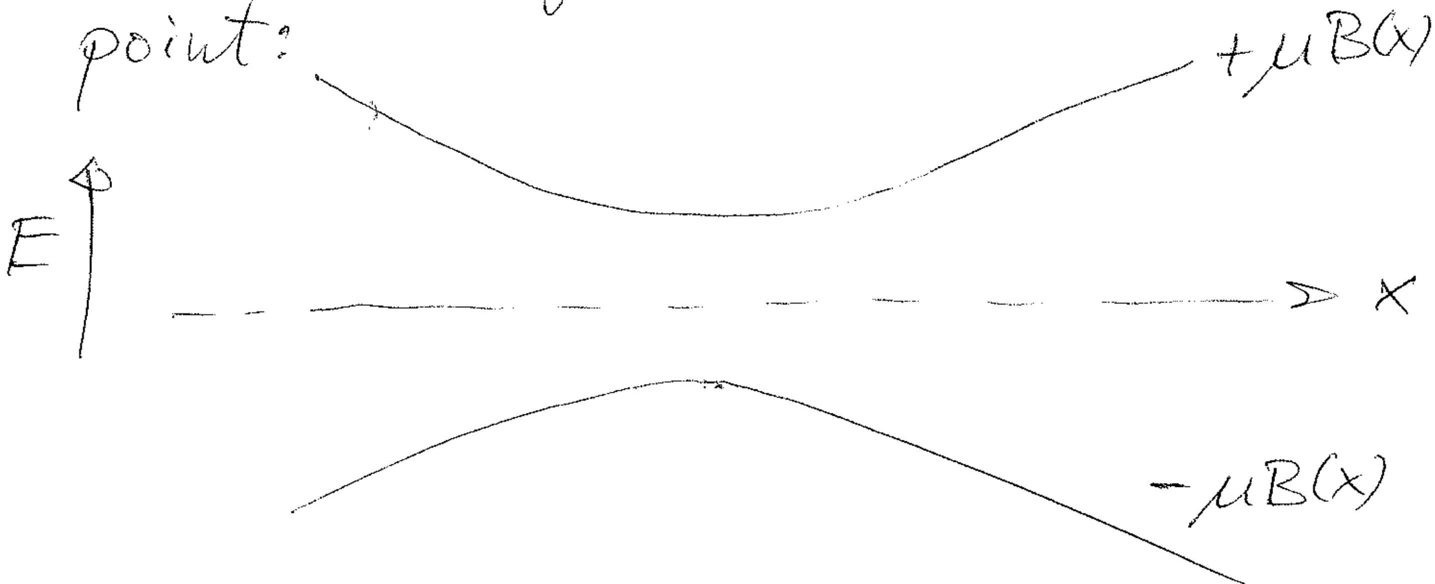
$$= -\left(\frac{\mu}{S}I\right)B(\vec{r})$$

Here $B(\vec{r})$ is the magnitude of the magnetic field. Previously, when the top was not adiabatically precessing but fixed in direction, this potential energy depended on $B_z(\vec{r})$. However, unlike $B_z(\vec{r})$ the function $B(\vec{r})$ does not satisfy the Laplace equation and it IS possible to construct magnetic fields where $B(\vec{r})$ has a local minimum and is stable in all three dimensions.

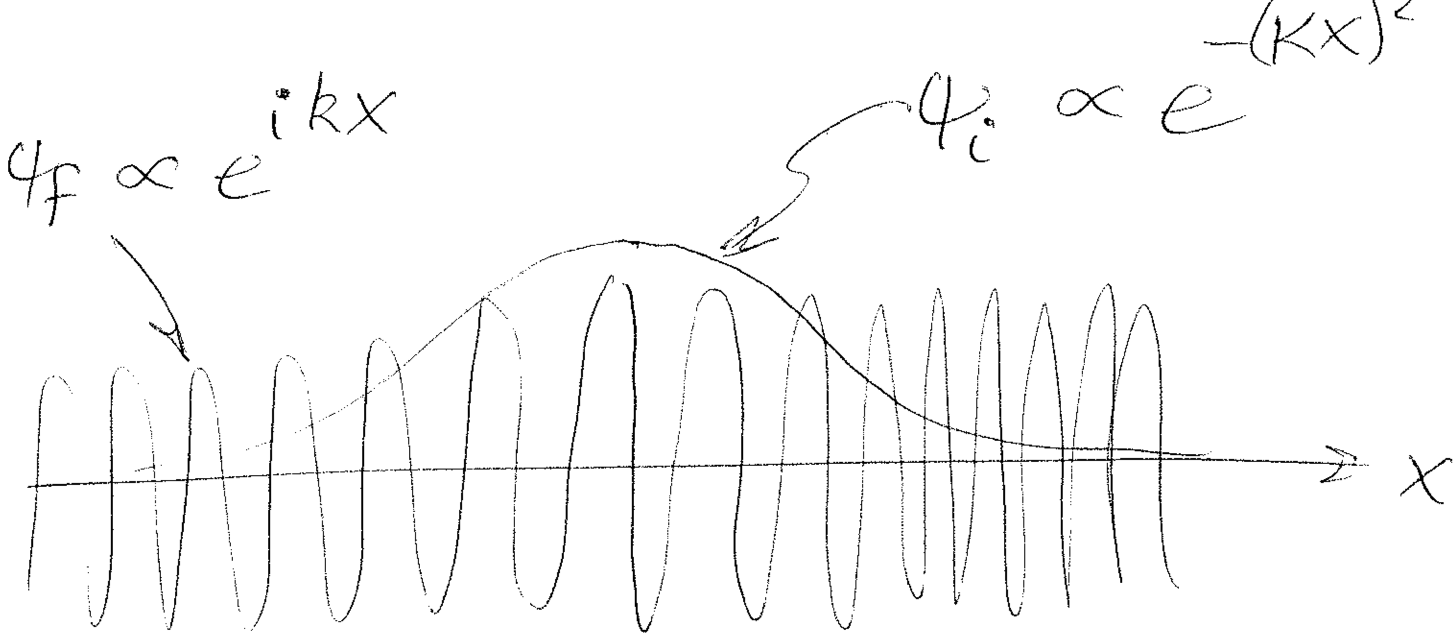
The stabilizing mechanism of the Levitron has a direct counterpart in the "magnetic traps" used to form cold gases ~~at~~. In place of the spinning top we have an atom with magnetic moment $\vec{\mu}$ parallel to its angular momentum $\vec{S}$, which has a fixed magnitude, say $\hbar/2$ (for spin-$1/2$). The action variable $\vec{S} \cdot \hat{b} = I$ takes two discrete values for states of definite energy, given by $I = \pm \hbar/2$. The dipole energy is then

$$\pm \mu B(\vec{r}),$$

which corresponds to two energy levels that depend on the <u>magnitude</u> of the magnetic field. The trap is designed so $B(\vec{r})$ increases away from the equilibrium point. The two energy levels would look ~~to~~ like this, along an x-axis that passes through the equilibrium point:



Adiabatic invariance in the context of this quantum particle is the statement that the system stays in one of the energy levels ($I = +\hbar/2$ or $I = -\hbar/2$) even while its position x is changing. We see that it is actually the <u>upper</u> energy level that traps the particle; particles in the lower level see a potential energy that causes x to escape to $\pm\infty$. But even particles trapped in the upper level have a small probability of making a quantum transition to the lower level, after which they will be expelled from the trap. A rough calculation of this transition propability is obtained by integrating the product of <u>initial</u> and <u>final</u> <u>wavefunctions</u> (ignoring the x-dependence of the perturbation) which look like this



$$\left.\begin{array}{l} \hbar\Omega \sim \dfrac{(\hbar K)^2}{2m} \\ \hbar\omega \sim \dfrac{(\hbar k)^2}{2m} \end{array}\right\} \Longrightarrow \frac{\Omega}{\omega} = \left(\frac{K}{k}\right)^2$$

$$(\text{trans. prob.}) \propto \left|\int dx\, \psi_f^* \psi_i\right|^2$$

$$\propto \left|\int dx\, e^{-(Kx)^2 + ikx}\right|^2$$

$$= \left| \int dx \; e^{-\left(Kx - \frac{i}{2}\frac{k}{K}\right)^2 - \frac{1}{4}\left(\frac{k}{K}\right)^2} \right|^2$$

$$\sim e^{-\frac{1}{2}\left(k/K\right)^2} = e^{-\frac{1}{2}\left(\omega/\Omega\right)^2}$$

This is exactly the "exponential-accuracy" of adiabatic invariance (i.e. no transition) we have come to expect.