Lec/Sec

P318 : the Hamiltonian

8 Mar

__Last time__: continuous symmetry of $L \rightarrow$ conserved quantity

eg homogeneity of time $\rightarrow$ H conserved

L does not dep explicitly on time. sym: $t \rightarrow t + \delta t$

energy (most of the time)

~~eg $L = \sum_{i=1}^{N} \frac{1}{2} m_i \dot{x}_i^2 + V(x_1, \ldots, x_N)$~~

Recap of the Lagrangian program

$L = T - V$ w/ EoM: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i}$

$m\ddot{x}_i = -\frac{\partial V}{\partial x_i}$ (CARTESIAN)

function of $(q_1, \ldots, q_N)$
and $(\dot{q}_1, \ldots, \dot{q}_N)$ } treat as independent (but they're not really!)

second O diff. eq.
[... & usually annoying!]

__Nice__: conserved quantities $\rightarrow$ can convert some to 1st O diff eq!

Hamiltonian dynamics

$(q_1, \cdots, q_N)$ as before

$(p_1, \cdots, p_N)$ $\quad p_i = \dfrac{\partial L}{\partial \dot{x}_i} = m_i \dot{x}_i$ (cartesian)

looks trivial...

eg CARTESIAN : $\quad H = \sum \frac{1}{2} p_i^2/m_i + V(x_1, \cdots, x_N)$

$$\frac{\partial H}{\partial x_i} = \frac{\partial V}{\partial x_i} \longleftrightarrow = -m_i \ddot{x}_i \equiv \dot{p}_i$$

(Euler-Lagr)

$$\frac{\partial H}{\partial p_i} = \frac{p_i}{m_i} \longleftarrow = \dot{x}_i$$

so: $$\boxed{\begin{aligned} \dot{p}_i &= -\partial H/\partial x_i \\ \dot{x}_i &= +\partial H/\partial p_i \end{aligned}} \Bigg\} \; 2N \; \underline{\underline{1^{st}\ \mathcal{O}}} \text{ eq!}$$

antisymmetric ("symplectic") structure

nice! Wouldn't it be great if it carried over to generalized coords?

$$H = \sum p_i \dot{q}_i - L \qquad \partial L/\partial \dot{q}_i$$

a duality transform, like FOURIER ("Legendre transform")

$$\frac{\partial H}{\partial p_j} = \dot{q}_j + \sum_{i=1}^{N} p_i \frac{\partial \dot{q}_i}{\partial p_j} - \frac{\partial L}{\partial p_j}$$

what we want ✪ where $\dot{q} = \dot{q}(q, p, t)$

$$\left.\frac{\partial L}{\partial p_j}\right|_q = \sum_{i=1}^{N} \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial p_j} = \sum_{i=1}^{N} p_i \frac{\partial \dot{q}_i}{\partial p_j}$$

FIX q ⇒ CANCELS ✪

$L = L(q, \dot{q})$ so have to include both.

$$\Rightarrow \boxed{\dot{q}_j = \frac{\partial H}{\partial p}}$$

next:
$$\left.\frac{\partial H}{\partial q_j}\right|_p = \sum_{i=1}^{N} p_i \frac{\partial \dot{q}_i}{\partial q_j} - \left.\frac{\partial L}{\partial q_j}\right|_p$$

not a func of q

$$\left.\frac{\partial L}{\partial q_j}\right|_{\dot{q}} + \sum \frac{\partial L}{\partial \dot{q}_i}\left.\frac{\partial \dot{q}_i}{\partial q_j}\right|_q$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) \qquad \sum p_i \left.\frac{\partial \dot{q}_i}{\partial q_j}\right|_p$$

EULER-LAGR.

cancel.

$$\boxed{\left.\frac{\partial H}{\partial q_j}\right|_p = \dot{p}_j}$$

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \to \dot{x} = \frac{p}{m}$$

<u>example</u>: 1D HO : $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$

$$H = p\dot{x} - L = p\dot{x} - \frac{1}{2}(m\dot{x}^2 - kx^2)$$

$$= p\left(\frac{p}{m}\right) - \frac{1}{2}m\left(\frac{p}{m}\right)^2 + \frac{k}{2}x^2$$

$$= \frac{p^2}{2m} + \frac{kx^2}{2} \qquad = E \checkmark$$

$$\dot{x} = \frac{\partial H}{\partial p} \longrightarrow \dot{x} = \frac{p}{m}$$

$$\dot{p} = -\frac{\partial H}{\partial q} \longrightarrow \dot{p} = -kx$$

<u>Easy to get</u>: $\frac{d}{dt}(m\dot{x}) = \dot{p} = -kx$ ✓

eg if we <u>started</u> w/ H.

Example problems

2D central force

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \longrightarrow \dot{r} = \frac{p_r}{m}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \longrightarrow \dot{\theta} = \frac{p_\theta}{mr^2}$$

$$\begin{aligned}
H &= \sum p_i \dot{q}_i - L \\
&= p_r \dot{r} + p_\theta \dot{\theta} - \left(\tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)\right) \\
&= p_r \frac{p_r}{m} + p_\theta \frac{p_\theta}{mr^2} - \frac{1}{2}m\left(\frac{p_r}{m}\right)^2 - \frac{mr^2}{2}\left(\frac{p_\theta}{mr^2}\right)^2 + V(r) \\
&= \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + V(r) = E \checkmark
\end{aligned}$$

unchanged

indep of θ $\longrightarrow$ "cyclic"

$$\dot{r} = \frac{\partial H}{\partial p_r} \longrightarrow \dot{r} = \frac{p_r}{m}$$

$$\dot{p}_r = -\frac{\partial H}{\partial r} \longrightarrow \dot{p}_r = \frac{p_\theta^2}{mr^3} - V'(r)$$

$$m\ddot{r} = \frac{p_\theta^2}{mr^3} - V'(r)$$

ℓ^2

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} \longrightarrow \dot{\theta} = \frac{p_\theta}{mr^2}$$

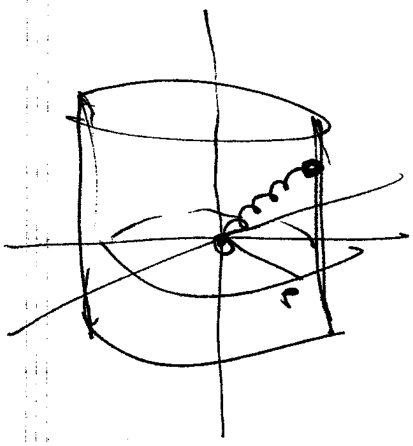
def $p_\theta = \ell$

conserved.

$$\dot{p}_\theta = \frac{-\partial H}{\partial \theta} \longrightarrow \dot{p}_\theta = 0 \checkmark$$

<u>example</u> particle on a cylinder

$x^2+y^2=R^2$ w/ SHO force wrt origin



$$V = \tfrac{1}{2}kr^2 = \underbrace{\tfrac{1}{2}k(\rho^2+z^2)}_{\text{cylindrical}}$$

$$T = \tfrac{1}{2}m\dot{\vec{r}}^2 = \tfrac{1}{2}m(R^2\dot{\theta}^2+\dot{z}^2)$$

USING $\rho = \text{CONST.} = R$

$$L = \tfrac{1}{2}m(R^2\dot{\theta}^2+\dot{z}^2) - \underbrace{\tfrac{1}{2}k}_{\text{const.}}(R^2+z^2)$$

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = mR^2\dot{\theta} \qquad p_z = \frac{\partial L}{\partial\dot{z}} = m\dot{z}$$

$$H = \frac{p_\theta^2}{2mR^2} + \frac{p_z^2}{2m} + \tfrac{1}{2}kz^2 + (\text{const})$$

$$\left.\begin{aligned}\dot{\theta} &= -\partial H/\partial p_\theta = p_\theta/mR^2\\ \dot{p}_\theta &= -\partial H/\partial\theta = 0\end{aligned}\right\} \rightarrow \ell \text{ conserved}$$

$$\left.\begin{aligned}\dot{z} &= \partial H/\partial p_z = p_z/m\\ \dot{p}_z &= -\partial H/\partial z = -kz\end{aligned}\right\} \ddot{z} = \tfrac{1}{m}\dot{p}_z = -\tfrac{1}{m}kz = -\omega^2 z \checkmark$$

eg. Hamilton eq. after a "GAUGE TRANSFORM"

$$\boxed{L' = L + \frac{d\Lambda}{dt}}$$ ← new L, differs by total time derivative

How does p' differ?

H' ?

EOM ?

$$L' = L + \sum \frac{\partial \Lambda}{\partial q_i}\dot{q}_i + \frac{\partial \Lambda}{\partial t}$$

$$\boxed{p'_i = \frac{\partial L'}{\partial \dot{q}_i} = \underbrace{\frac{\partial L}{\partial \dot{q}_i}}_{p} + \underbrace{\frac{\partial \Lambda}{\partial q_i}}_{\text{new}}}$$

$$H' = \sum p'\dot{q} - L' = \sum\left(p_i + \frac{\partial \Lambda}{\partial q_i}\right)\dot{q}_i - L - \sum \frac{\partial \Lambda}{\partial q_i}\dot{q}_i - \frac{\partial \Lambda}{\partial t}$$

$$= H\left(q, \underbrace{p' - \frac{\partial \Lambda}{\partial q}}_{\text{compare to E\&M}}\right) - \frac{d\Lambda}{dt}$$

("gauge transform")

Hamilton eq:

$$\dot{q}_i = \frac{\partial H'}{\partial p'_i} = \frac{\partial H}{\partial p}$$

from above

$$\dot{p}' = -\left(\frac{\partial H'}{\partial q_i}\right)$$

H(p')

$$\frac{d}{dt}\left(p_i + \frac{\partial \Lambda}{\partial q_i}\right) = -\frac{\partial}{\partial q_i}\left(H - \frac{d\Lambda}{dt}\right)\Big|_{p'}$$

$$\dot{p}_i + \sum_j \frac{\partial^2 \Lambda}{\partial q_i \partial q_j}\dot{q}_j + \cancel{\frac{\partial \Lambda}{\partial t}} \frac{d}{dt}\frac{\partial \Lambda}{\partial q_i}$$

$$= -\frac{\partial H}{\partial q_i} + \sum_j \frac{\partial H}{\partial p_j}\Big|_{q_i} \underbrace{\frac{\partial p'_j}{\partial q_i}}_{\frac{\partial^2 \Lambda}{\partial q_i \partial q_j}} + \frac{d}{dt}\frac{\partial \Lambda}{\partial q}$$

$$\Rightarrow \dot{p} = -\frac{\partial H}{\partial q_i} \quad \checkmark$$

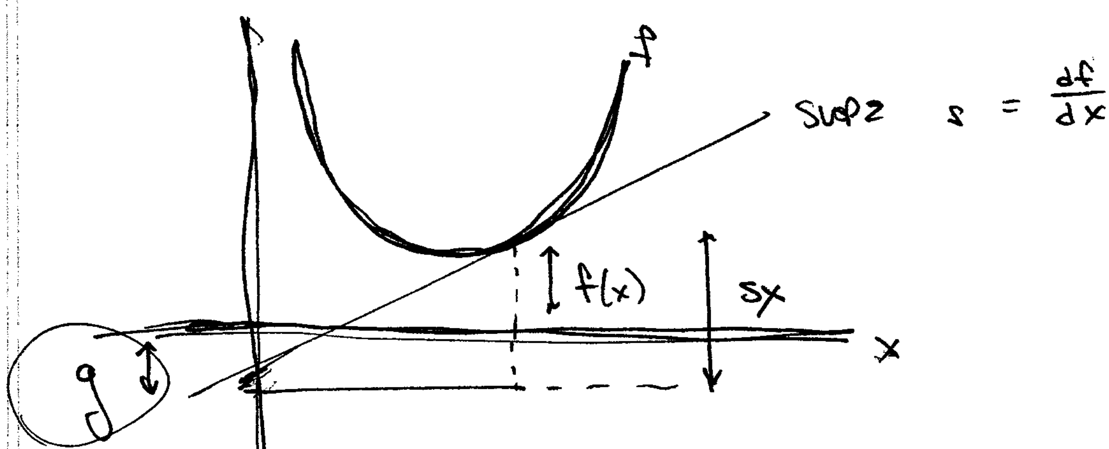
Meaning of Legendre transform

$L \to H$ is most trivial example
but this shows up over & over again
in physics!

Duality transform: like fourier transform

(Legendre) × (Legendre) = 1

req: sufficiently smooth
convex



intercept
function
of slope
& f(x)

$$\boxed{g(s) + f(x) = sx}$$