Assignment 9

1. H&F 7.5
2. H&F 7.6
3. H&F 7.13
4. H&F 7.14
5. H&F 8.3 For the physical tensor, just find the principal moments, not the axes.
6. H&F 8.10

Problem 2: *($\vec{\omega}$ as pseudovector)* How does the matrix **A** (or, equivalently, the vector $\vec{\omega}$) transform if all coordinates are reversed as under a space reflection? Prove that $\vec{\omega}$ transforms like a pseudovector rather than a true vector. (A vector reverses sign under a spatial reflection; a pseudovector does not.)

Problem 3: *(Rolling cone)* A cone rolls on a flat surface. The instantaneous axis of rotation lies parallel to the point where the cone touches the surface and the angular velocity is $\vec{\omega}$. The motion consists of a motion of the center of mass ($\vec{V}_{cm}$) plus a rotation $\vec{\omega}_{cm}$ about the center of mass. Describe this motion by finding $\vec{V}_{cm}$ and $\vec{\omega}_{cm}$ in the laboratory (space) system.

Problem 4: *(Rolling sphere)* A sphere of radius R rolls without slipping on a flat surface with angular velocity $\vec{\omega}$. Since rolling without slipping means that the velocity of the point of tangency between the sphere and the surface is zero, this gives a relation between $\vec{V}_{cm}$ and $\vec{\omega}$. Find this relation, which is a constraint on the motion. How many degrees of freedom does the sphere have?

Problem 5: *(Charged electron)* In some respects an electron is like a charged spinning top. The electron has internal angular momentum and a magnetic moment, so it behaves like a magnetic dipole oriented along the spin axis. In a magnetic field the equation of motion for the spin angular momentum $\vec{\sigma}$ in a magnetic field $\vec{B}$ is

$$\dot{\vec{\sigma}} = g'(\vec{\sigma} \times \vec{B}), \tag{7.65}$$

where $g' \equiv \frac{e\hbar}{2mc} g$, e is the electronic charge, $\hbar$ Planck's constant divided by 2π, m the electron's mass, and c the velocity of light. The constant g is called the "gyromagnetic ratio." For electrons, $g \approx 2$.

a) Show that in a frame rotating at a certain angular velocity, the effect of the magnetic field can be made to vanish. Find this angular velocity $\vec{\omega}_0$.
b) Suppose the magnetic field has two components: $\vec{B} = B_0 \hat{k} + \vec{B}_1$, where B_0 is a constant, $\hat{k}' = \hat{k}$ is a unit vector in the Z', Z direction, and $\vec{B}_1$ is a rotating magnetic field of constant magnitude: $\vec{B}_1 = B_1(\cos\omega t\, \hat{\imath}' + \sin\omega t\, \hat{\jmath}')$. Regarding ω as a variable parameter, find the equation of motion in a frame rotating with angular velocity ω and solve it. Describe qualitatively what happens to the spin if a) $\omega = \omega_0$, b) $\omega \neq \omega_0$. (This is the basic equation of NMR – nuclear magnetic resonance – except that the spin is that of an atomic nucleus, not of an electron.)

Orthogonal Matrices

Problem 6: *(The most general form)*

a) Find the most general form of an orthogonal 2×2 matrix. What is the geometric interpretation of such a matrix? What are the complex eigenvalues of such a matrix? What special property do they have? What is the determinant of the general 2-D orthogonal matrix?

b) Write out the 3×3 separate matrices for 90° clockwise rotations about the X, Y, and Z axes. Find the products of these rotation matrices about Z first, Y next, and X last. Then find the product for rotating in the reverse order. Interpret the result in terms of the experiment with the book in Section 7.3. (The corresponding questions from part a) for 3×3 orthogonal matrices are more difficult to answer. We will develop an explicit form for the 3-D orthogonal matrix $\mathbf{U}$ in terms of the three Euler angles in the next chapter.)

Problem 7*: *(General properties of orthogonal matrices)* This problem involves proving some general properties of orthogonal matrices in a space of arbitrary dimensions. It will be necessary to know some facts about determinants that hold for any arbitrary $n \times n$ matrix $\mathbf{M}$:

$$\det \mathbf{M} = \det \tilde{\mathbf{M}}, \quad \det(-\mathbf{M}) = (-1)^n \det \mathbf{M}, \tag{7.66}$$

$$\det(\mathbf{AB}) = \det(\mathbf{BA}) = \det \mathbf{A} \det \mathbf{B}. \tag{7.67}$$

You may wish to review the derivations of (7.66, 7.67) in a book on linear algebra.[†] Use the above identities to prove the following:

a) If $\mathbf{U}$ is a real orthogonal $n \times n$ matrix, prove

$$\det \mathbf{U} = \pm 1. \tag{7.68}$$

(If $\det \mathbf{U} = 1$, $\mathbf{U}$ is a proper rotation; if $\det \mathbf{U} = -1$, $\mathbf{U}$ is an improper rotation, i.e., a reflection plus a proper rotation.)

b) For a proper rotation of any odd-dimensionality n, prove that the orthogonal matrix $\mathbf{U}$ has at least one eigenvalue equal to 1; hence there is an "axis" of rotation – a direction that is invariant under the transformation $\mathbf{U}$. (Hint: First prove that if there is an eigenvalue equal to 1, $\det(\mathbf{U} - \mathbf{1}) = 0$, where $\mathbf{1}$ is the identity matrix.)

c) The trace of a matrix is the sum of its diagonal elements. Prove that the trace of any matrix $\mathbf{M}$ is invariant under an orthogonal transformation: $\mathbf{M}' = \mathbf{U}\mathbf{M}\tilde{\mathbf{U}}$. (Hint: $\text{Trace}(\mathbf{AB}) = \text{Trace}(\mathbf{BA})$. Prove this first; then prove that $\text{Trace}(\mathbf{M}') = \text{Trace}\,\mathbf{M}$.)

Problem 8*: *(Trace of U)* If $\mathbf{U}$ is a real orthogonal 3×3 matrix, show that the trace of $\mathbf{U}$ equals

$$\text{Trace}\,\mathbf{U} = 1 + 2\cos\Phi, \tag{7.69}$$

where Φ is the angle of rotation. (Hint: We proved in Problem 7c that the trace is invariant when the basis vectors are changed to a new set by an orthogonal transformation. Try moving the axis of rotation to the Z axis by such a transformation.)

[†] For example: *Linear Algebra with Applications*, 2d ed., by Steven J. Leon.

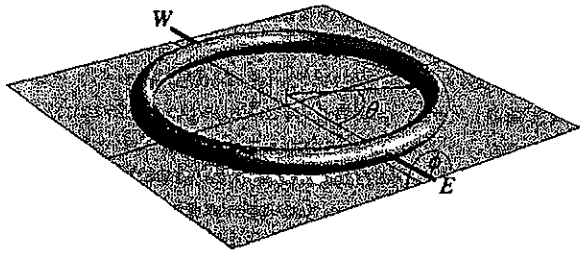


FIGURE 7.15
Compton generator.

the effect of gravity. How does this change the result? Use $\omega_{earth} = 7.3 \times 10^{-5}$ radians/sec.

Problem 12: *(Deflection of object thrown up into the air)* A heavy object is thrown up into the air. Calculate the deflection of the object when it hits the ground due to the Coriolis force. Compare the results to those of an object dropped at rest from its maximum height.

Problem 13: *(Compton generator)* When he was an undergraduate, the famous physicist A. H. Compton invented a simple way to measure the rotation of the Earth with a table-top experiment. The "Compton generator," as it was called, is a circular hollow glass tube shaped like a doughnut as shown in Figure 7.15. The inside of the tube is filled with water. Imagine that the "doughnut" lies flat on a table and is then turned over by rotating it 180° around a diameter, such that it again lies flat on the table surface, which is horizontal. The result of the experiment is that the water moves with a certain constant drift velocity around the tube after the doughnut has been rotated. If there were no friction with the walls, the water would continue to circulate indefinitely.

a) Prove that the axis about which the doughnut is flipped should be oriented east–west to maximize the drift velocity of the water. What will happen if it is oriented north–south instead?
b) Let θ be defined as the angle between a small volume of the water and the "flip" axis. Calculate the component of the Coriolis force *parallel* to the wall of the circular tube, F_θ, while the circular tube is being flipped 180°. For simplicity, assume it has been flipped through 90° already and is still moving. Draw little arrows for different positions in the tube (different θ values) to show the relative magnitude of the tangential component of the Coriolis force at that point at time. Why don't we have to consider the radial component of this force?
c) Let the angle of rotation about the diameter be ϕ. Then ϕ starts at zero, and when the tube has been flipped over, $\phi = 180°$. From the time integral of the tangential force, calculate the change in the total tangential momentum of the water in the tube. This means integrating over θ around the rim of the tube. Show that the total tangential momentum does not depend on the $\phi(t)$ but only on the total change in

ϕ. Evaluate this total momentum for a change of 180° in ϕ. You can assume the water is equidistant from the center of the circle at some constant radius R. Does the water circulate clockwise or counterclockwise?

d) Since water is incompressible, the water molecules in the tube must all drift with the same velocity after the tube is flipped. With this assumption, you can calculate this drift velocity as a function of the Earth's angular velocity and the latitude. Prove that

$$v_{\text{drift}} = 2\omega R \sin\lambda. \tag{7.71}$$

Compton used small droplets of coal oil mixed in the water to measure the drift velocity under a microscope. The experiment consists of laying the tube flat on a table until the water in it came to equilibrium, then slowly (in about 3 seconds) rotating it about an east–west axis until it had turned 180° and was again lying flat on the table.

e) Compton used this measured drift velocity to determine his latitude. His $\sin\lambda$ was measured to within 3% accuracy, which is pretty good for such a simple device. Assuming $\lambda = \frac{\pi}{4}$ and $R = 1$ meter, what is v_{drift} in mm/s?

Problem 14: *(Force-free motion as seen from a turntable)* Plot the trajectory of the force-free motion of a particle as seen from a frame rotating with a constant angular velocity ω. Assume the particle starts with an initial outward radial velocity v_0, at an initial position halfway towards the rim. If R is the turntable radius, let

$$v_0 = f\omega R. \tag{7.72}$$

Plot this motion for $f = 0, 0.5, 0.7,$ and 3. Notice that all trajectories are straight lines in an inertial frame.

Problem 15: *(Hurricanes)* Prove that the steady-state motion of the wind near a low pressure area in the atmosphere is a circle along the lines of constant pressure if air resistance is neglected. Also: Why are there no hurricanes near the equator?

Problem 16: *(Foucault pendulum)* Here is a different method for solving the Foucault pendulum. Assume the motion of the pendulum is given by

$$\begin{aligned} x(t) &= A_x(t)\cos(\omega_P t + \Phi_x(t)), \\ y(t) &= A_y(t)\cos(\omega_P t + \Phi_y(t)), \end{aligned} \tag{7.73}$$

where $\omega_P \equiv \sqrt{\frac{g}{l}}$ is the pendulum frequency. In the absence of Coriolis and centrifugal "forces" $A_{x,y}$, $\Phi_{x,y}$ would be arbitrary constants. If these "forces" are present, they become slowly changing functions of the time. Insert the expressions (7.73) into the

Problem 2: *(Orthogonal transformation represented as rotation)* Prove that any orthogonal transformation **U** can be represented as a rotation though an angle about an axis that is left unchanged by **U**.

Moment of Inertia Tensor

Problem 3: *(Moment of inertia tensor)* Which of the symmetric 3×3 matrices below could represent a physical moment of inertia tensor?

$$I_1 = \begin{pmatrix} 1 & 2 & 1 \\ \dots & 0 & 2 \\ \dots & \dots & 1 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1.94791 & .0347273 & -.394509 \\ \dots & 2.42924 & -.823746 \\ \dots & \dots & 1.62285 \end{pmatrix} \tag{8.131}$$

Explain. Find the principal axes and principal moments of the ones that are physical.

Problem 4: *(I for a circular hoop)* What is the moment of inertia tensor for a circular hoop of radius R and mass M? What are the principal axes and moments? (Neglect the thickness of the hoop.)

Problem 5: *(I for a thin rod)* Find the inertia tensor, principal axes, and principal moments for a thin rod of length l.

Problem 6: *(I for a circular cylinder)* Find the inertia tensor, principal axes, and principal moments for a circular cylinder of radius R and height h.

Problem 7: *(I for an ellipsoid)* Find the inertia tensor, principal axes, and principal moments for an ellipsoid of semiaxes a, b, c.

Problem 8: *(I for a spherical shell, solid sphere)* Calculate the moment of inertia tensor for a spherical shell of radius R and mass M. Simplify your calculation by using the symmetry to maximum advantage. From this result calculate **I** for a solid sphere of radius R.

Problem 9*: *(I for three mass points)* Three equal mass points are located at $(a, 0, 0)$, $(0, a, 2a)$, and $(0, 2a, a)$. Find the inertia tensor, the principal axes, and the principal moments.

Problem 10*: *(I for a book)* A book of mass M has the dimensions $a = 10\,\text{cm}$ by $b = 20\,\text{cm}$ by $c = 3\,\text{cm}$. Find the principal axes using a symmetry argument. Find the inertia tensor in the principal axis system. Indicate on a diagram the direction of the principal axes and which ones have the least moment of inertia and the greatest moment of inertia.

P3318 HW9 SOLUTIONS

DUE 28 APRIL

1. (H&F 7.5) CHARGED ELECTRON

spin $\vec{\sigma}$, magnetic fied $\vec{B}$: $\dot{\vec{\sigma}} = g'(\vec{\sigma} \times \vec{B})$

$g' = \frac{e\hbar}{2mc} \cdot g$ $\qquad$ $g \equiv$ gyromagnetic constant

A: IN A ROTATING FRAME AT CERTAIN $\vec{\omega}_0$, $\vec{B}$ CAN BE MADE TO DISAPPEAR:

$$\dot{\vec{\sigma}}_{fixed} = \dot{\vec{\sigma}}_{rot} + \vec{\omega} \times \vec{\sigma} \Rightarrow g'(\vec{\sigma} \times \vec{B}) + \vec{\omega} \times \vec{\sigma} = \dot{\vec{\sigma}}_{rot}$$

$$\dot{\vec{\sigma}}_{rot} = (\vec{\omega} + g'\vec{B}) \times \vec{\sigma}$$

IF $\boxed{\vec{\omega} = \vec{\omega}_0 = -g'\vec{B}}$ $\vec{B}$ disappears.

B: $\vec{B} = B_0 \hat{k} + \vec{B}_1$ $\qquad$ where $\vec{B}_1 = B_1 (\cos(\omega t)\hat{\imath}' + \sin(\omega t)\hat{\jmath}')$

FIND EOM IN A FRAME ROTATING AT ω AND SOLVE IT.

In a rotating frame with $\vec{\omega} = \omega \hat{k}$

$$\vec{B} = B_0 \hat{k} + B_1 \hat{\imath}$$

And $\qquad \dot{\vec{\sigma}} = (\vec{\omega} + g'\vec{B}) \times \vec{\sigma} = \left(g'B_1 \hat{\imath} + (\omega + g'B_0)\hat{k}\right) \times \vec{\sigma}$

$$\dot{\vec{\sigma}} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ g'B_1 & 0 & \omega + g'B_0 \\ \sigma_x & \sigma_y & \sigma_z \end{vmatrix} = \left(-(\omega + g'B_0)\sigma_y,\ -g'B_1\sigma_z + (\omega + g'B_0)\sigma_x,\ g'B_1\sigma_y\right)$$

$$\left.\begin{aligned}\dot{\sigma}_x &= -(\omega + g'B_0)\sigma_y \\ \dot{\sigma}_y &= -g'B_1\sigma_z + (\omega + g'B_0)\sigma_x \\ \dot{\sigma}_z &= g'B_1\sigma_y\end{aligned}\right\} \quad \ddot{\sigma}_y = -g'B_1\dot{\sigma}_z + (\omega + g'B_0)\dot{\sigma}_x$$

$$\ddot{\sigma}_y = -(g'B_1)^2\sigma_y - (\omega + g'B_0)^2\sigma_y = -\Omega^2\sigma_y$$

where $\Omega^2 = (g'B_1)^2 + (\omega + g'B_0)^2$

$\sigma_y = A\cos(\Omega t + \phi)$ where A and ϕ are given by the initial conditions.

$$\dot{\sigma}_x = -(\omega + g'B_0)A\cos(\Omega t + \phi)$$

$$\sigma_x = -A\frac{(\omega + g'B_0)}{\Omega}\sin(\Omega t + \phi)$$

$$\dot{\sigma}_z = g'B_1 A\cos(\Omega t + \phi) \Rightarrow \sigma_z = A\frac{g'B_1}{\Omega}\sin(\Omega t + \phi)$$

$$\boxed{\vec{\sigma} = \frac{A}{\Omega}\Big((\omega + g'B_0)\sin(\Omega t + \phi), \Omega\cos(\Omega t + \phi), g'B_1\sin(\Omega t + \phi)\Big)}$$

IT IS A PRECESSION MOVEMENT.

In the special case where $\omega = \omega_0 \Rightarrow \Omega = g'B_1$

and $\vec{\sigma} = \frac{A}{\Omega}(0, \cancel{\Omega}\cos(\Omega t + \phi), \sin(\Omega t + \phi))\Omega = A(0, \cos(\Omega t + \phi), \sin(\Omega t + \phi)$

which is a rotation around the x-axis.

2 (H&F 7.6) THE MOST GENERAL FORM

A: MOST GENERAL ORTHOGONAL 2x2 MATRIX

Orthogonal matrix: $Q^T = Q^{-1} \Rightarrow Q\cdot Q^T = Q^T\cdot Q = I$

$$2\times2: \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+c^2 & ab+cd \\ ab+cd & a^2+b^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{system: } \begin{cases} a^2+b^2=1 \\ a^2+c^2=1 \\ ab+cd=0 \\ ac+bd=0 \end{cases} \quad \text{SOLUTIONS: } Q_1 = \begin{pmatrix} a & -\sqrt{1-a^2} \\ \sqrt{1-a^2} & a \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$Q_2 = \begin{pmatrix} -a & \sqrt{1-a^2} \\ \sqrt{1-a^2} & a \end{pmatrix} = \begin{pmatrix} -\cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Q_1 ARE ROTATION MATRICES	Q_2 ARE REFLECTION MATRICES

EIGENVALUES: $|Q_1 - \lambda I| = (a-\lambda)^2 + (1-a^2) = 0 \Rightarrow \lambda = \cos\theta \pm i\sin\theta = e^{\pm i\theta}$

$|Q_2 - \lambda I| = -(a-\lambda)(a+\lambda) - (1-a^2) = 0 \Rightarrow \lambda = \pm 1$

Eigenvalues must have modulus 1.

DETERMINANT: $1 = |I| = |Q\cdot Q^T| = |Q||Q^T| = |Q|^2 \Rightarrow \boxed{|Q| = \pm 1}$

B: $R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$ $\quad R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$

$R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$R_x(\pi/2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad R_y(\pi/2) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$R_z(\pi/2) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_x(\pi/2)\,R_y(\pi/2)\,R_z(\pi/2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$R_z(\pi/2)\,R_y(\pi/2)\,R_x(\pi/2) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} =$$

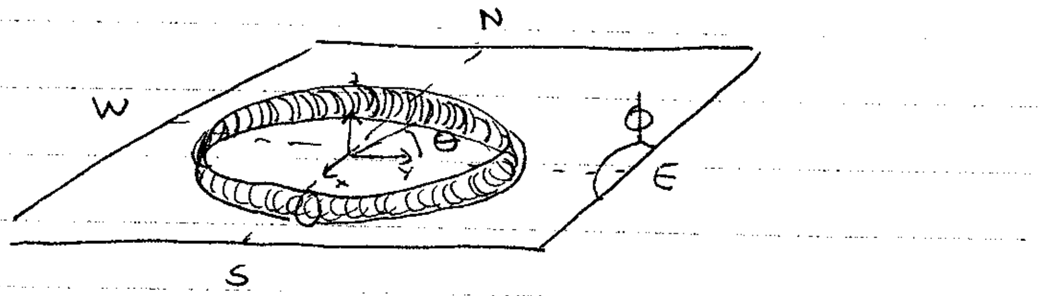
$$= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

INTERPRETATION: ROTATIONS IN GENERAL <u>DO NOT</u> COMMUTE.

In fact, for a general angles in 3-dimensions we obtain the relation:

$$R_x(\phi)\,R_y(\theta)\,R_z(\psi) = R_z^T(\psi)\,R_y^T(-\theta)\,R_x^T(-\phi)$$

3. (H&F 7.13) COMPTON GENERATOR



We choose a coordinate system where $\hat{z} \equiv \hat{R}$ points upwards in the direction of the radial vector $\vec{r}$ from the surface of the Earth.
We choose $\hat{j}$ pointing East and $\hat{i}$ pointing South.

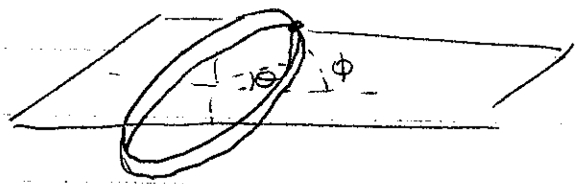
In these coordinates the angular velocity of the Earth is:

$$\vec{\omega} = (-\omega\cos\lambda, 0, \omega\sin\lambda)$$

A The Coriolis force is proportional to $\vec{v} \times \vec{\omega}$ and therefore for a rotation around a general ~~vert~~ axis the N-S components have different signs symmetrically and vanish.
Therefore a rotation around the N-S axis will produce no velocity on the water, and a rotation around the E-W axis maximizes the velocity.
(

B

The position of a given pont in the ring is:



$$\vec{r} = (R\cos\theta\cos\phi,\ R\sin\theta,\ R\cos\theta\sin\phi)$$

Which means that it has a velocity:

$$\vec{v}_r = (-R\cos\theta\sin\phi\,\dot{\phi},\ 0,\ R\cos\theta\cos\phi\,\dot{\phi})$$

And so the Coriolis force acting on it is:

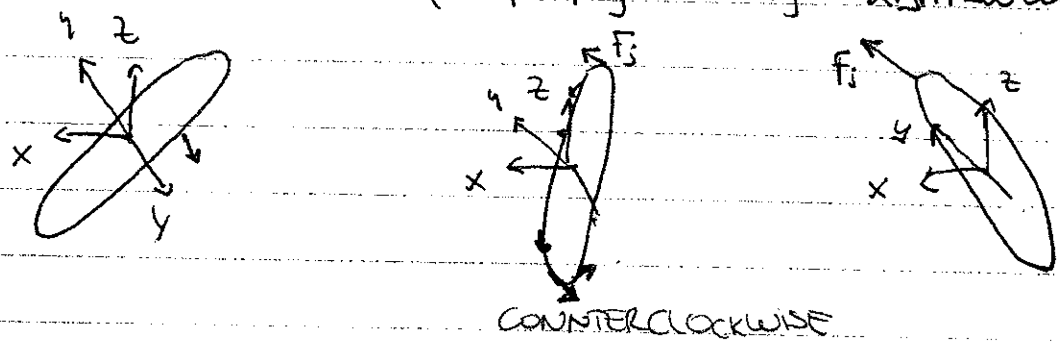
$$\vec{\omega}\times\vec{v}_r = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -\omega\cos\lambda & 0 & \omega\sin\lambda \\ -R\cos\theta\,\dot{\phi}\sin\phi & 0 & R\cos\theta\cos\phi\,\dot{\phi} \end{vmatrix}$$

$$= (R\omega\cos\lambda\cos\theta\cos\phi\,\dot{\phi} - R\omega\cos\theta\sin\lambda\sin\phi\,\dot{\phi})\,\hat{\jmath}$$

$$= R\omega\cos\theta\,(\cos\lambda\cos\phi - \sin\lambda\sin\phi)\,\dot{\phi}\,\hat{\jmath}$$

$$= R\omega\cos\theta\cos(\lambda+\phi)\,\dot{\phi}\,\hat{\jmath}$$

$$F_j = -2\Delta m R\omega\cos\theta\cos(\lambda+\phi)\,\dot{\phi}\,\hat{\jmath}$$



$$F_\theta = -2\Delta m F_j\cos\theta = -F2\Delta m\,\omega\cos\theta\cos(\lambda+\phi)\,\dot{\phi}\,\hat{U}_\theta$$

$$F_r = -2\Delta m F_j\sin\theta = -F\Delta m\,\omega\sin 2\theta\cos(\lambda+\phi)\,\dot{\phi}\,\hat{U}_r$$ will vanish upon integration of θ from 0 to 2π.

C $\quad F_\theta = \frac{dP_\theta}{dt}$

First we evaluate the total tangential force; integrating over θ

$$F_{\theta_{tot}} = -2\int_0^{2\pi} R\omega \cos(\lambda+\phi)\dot{\phi} \cos^2\theta \, \hat{u}_\theta \, \Delta m$$

$$= 2\pi R\omega \cos(\lambda+\phi)\dot{\phi}\, \Delta m$$

$$\Rightarrow \frac{dP_\theta}{dt} \propto \cos(\lambda+\phi)\cdot\dot{\phi} = \cos(\lambda+\phi)\frac{d\phi}{dt} \Rightarrow dP_\theta \propto d\phi$$

And so P_θ cannot depend on $\dot{\phi}$.
Considering a change of $180°$ in ϕ we would get a total momentum:

$$P_{\theta_t} = -2\pi R\omega \Delta m \int_0^{\pi} \cos(\lambda+\phi)d\phi$$

$$= -2\pi R\omega \Delta m \left[\sin(\lambda+\pi) - \sin\lambda\right]$$

$$= 4\pi R\omega \sin(\lambda)\Delta m$$

D The mean momentum for the particles in the tube is

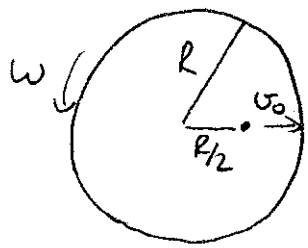
$$\langle P_\theta \rangle = \frac{P_{\theta_t} M}{2\pi \Delta m} = \cancel{4}\; 2R\omega \sin\lambda \cdot m = m \cdot v_{drift}$$

And so the drift velocity is $\boxed{v_{drift} = 2R\omega \sin\lambda}$

E $\quad \lambda = \pi/4 \quad R = 1\,m \quad \omega = \frac{2\pi}{T} = \frac{2\pi}{1\,day} = 7'3\cdot 10^{-5}\,rad$

$$v_{drift} = \cancel{2}\cdot 7'3\cdot 10^{-5}\cdot \frac{\sqrt{2}}{\cancel{2}}\ m/s = 0'1\ mm/s$$

4 (H&F 7.14). FORCE FREE PARTICLE AS SEEN FROM A TURNTABLE



$r_0 = R/2$

$v_0 = f\cdot \omega R \qquad f = 0,\ 0'5,\ 0'7,\ 3$

↑ "coriolis parameter"

In an inertial frame:

$$\dot{\vec r} = f\omega R\,\hat u_r = \text{constant}$$

$$\dot{\vec r} = f\omega R(\cos\theta_0, \sin\theta_0) \Rightarrow \vec r = (R/2\cos\theta_0 + f\omega R\cos\theta_0\, t)\hat\imath + (R/2\sin\theta_0 + f\omega R\sin\theta_0\, t)\hat\jmath$$

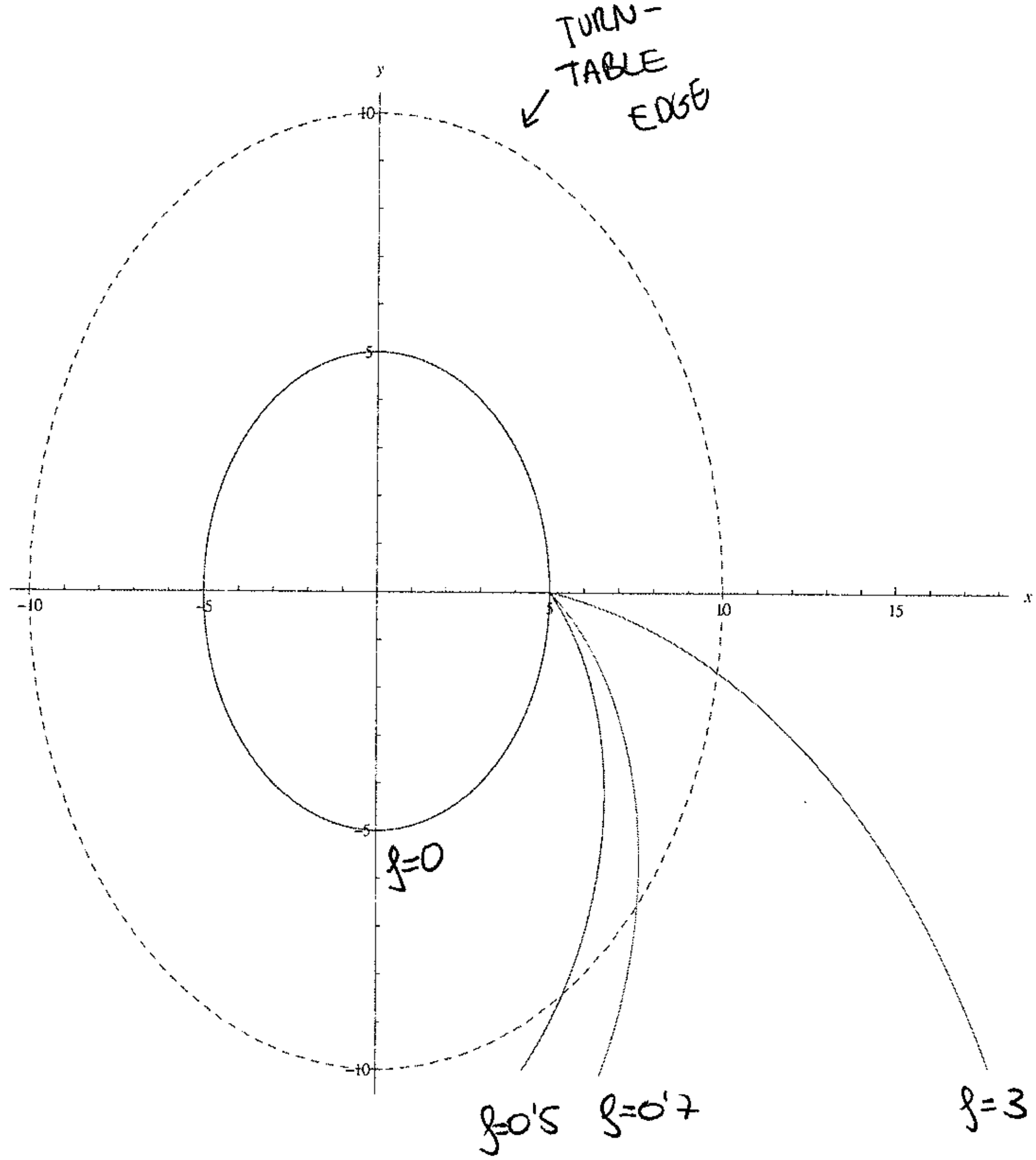
We pick $\theta_0 = 0$

$\Rightarrow \vec r = (R/2 + f\omega R t)\hat\imath$ in an inertial frame.

In a rotating reference frame at ω:

$$\vec r = (R/2 + f\omega R t)\cos\omega t\,\hat\imath + (R/2 + f\omega R t)\sin\omega t\,\hat\jmath$$

$$\begin{cases} x = (R/2 + f\omega R t)\cos\omega t \\ y = (R/2 + f\omega R t)\sin\omega t \end{cases}$$


TURN-
TABLE
EDGE
y
10
5
-5
-10
-10
-5
5
10
15
x
f=0
f=0'5
f=0'7
f=3

5. (H & F 8.3) Moment of Inertia tensor

We are given the upper right half of two moment of inertia tensors.

I. We know the <u>entire</u> tensor because I is symmetric:

$$I_{ij} = I_{ji}$$

II. Positivity of kinetic energy tells us that the eigenvalues of I are non-negative:

$$I_a \geqslant 0 .$$

a) FIND EIGENVALUES λ VIA SOLUTIONS TO THE CHARACTERISTIC EQUATION

$$|I - \lambda \mathbb{1}| = 0$$

[EASIEST WAY: plug into Mathematica]

$\Rightarrow$ NEGATIVE EIGENVALUE $\Rightarrow$ $\boxed{\text{not a good mom. of inertia tensor}}$

b) similarly for I_2:

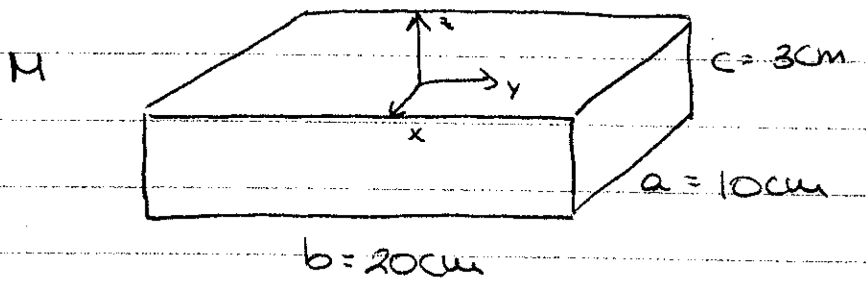
[plug into Mathematica]

the eigenvalues are all non-negative

↳ $\boxed{\text{VALID MOMENT of INERTIA}}$

$$I_a = \lambda = \boxed{1, 2, 3}$$

6. (H&F 8.10) I FOR A BOOK



For symmetry arguments the principal axes are the ones along the axes of the book (a, b, c) with the origin placed at the center of mass.

We should notice that, given an axis, only the points of V along the diagonal of the plain perpendicular to that axis will contribute. All the other points have symmetric points with opposite contributions. Therefore, if $dM = \rho dV$

$$I_z = \int \rho(\vec{r})\vec{r}^2 dV = \int \rho \cdot z^2 dxdydz$$

$$= \rho \cdot \int_{-c/2}^{c/2} dz \int_{-b/2}^{b/2} dy \int_{-a/2}^{a/2} dx \cdot (x^2+y^2) = \rho \cdot \frac{abc}{12}(a^2+b^2) = \frac{M}{12}(a^2+b^2$$

Symmetrically: $I_x = \frac{M}{12}(b^2+c^2)$ $I_y = \frac{M}{12}(a^2+c^2)$

Numerically $I_x = \frac{M}{12} 409$ $I_y = \frac{M}{12} 109$ $I_z = \frac{M}{12} 500$

So $I_z > I_x > I_y$ just as $c < a < b$.