Assignment 3

Due date: Wednesday, February 13

1. H&F 2.6
2. H&F 2.9
3. H&F 2.10
4. H&F 2.16
5. The action functional for a 1D harmonic oscillator is

$$S[x(t)] = \int_0^T \left(\tfrac{1}{2}m\,\dot{x}^2 - \tfrac{1}{2}m\omega_0^2\,x^2\right) dt,$$

and the trajectory endpoints are fixed as

$$x(0) = x_1, \qquad x(T) = x_2.$$

In this problem you will study arbitrary trajectories when expressed in the form

$$x(t) = \tilde{x}(t) + \delta x(t),$$

where $\tilde{x}(t)$ is an extremal trajectory given by Hamilton's principle (and therefore satisfies the Euler-Lagrange equation) and $\delta x(t)$ is whatever is left over and is **not assumed to be small.**

(a) Show that $S[x(t)] = S[\tilde{x}(t)] + \delta S$, where

$$\delta S = \int_0^T \left(\tfrac{1}{2}m\,\delta\dot{x}^2 - \tfrac{1}{2}m\omega_0^2\,\delta x^2\right) dt.$$

Since $\delta x(0) = \delta x(T) = 0$, consider perturbations having the form

$$\delta x(t) = \Delta \sin\left(N\pi t/T\right),$$

where Δ is the amplitude of the perturbation and the integer N counts the number of wiggles between the endpoints.

(b) Using the above form for $\delta x(t)$, show that $\delta S = c_N\Delta^2$, and determine the constant c_N. Further, show that for the case $\omega_0 T > \pi$ (trajectories that span more than one half-period), c_N can have either sign, depending on N. The action functional thus will not always be a simple minimum or maximum at the extremum, but more generally, a "saddle" having both signs of curvature, depending on direction.

that rolling up the paper does not change the geodesic property of the curves on the surface. Why not? Hint: Set up a suitable coordinate system, and find an integral expression for the length of an arbitrary curve on the surface of this cylinder.

Problem 4: *(Geodesic on a cone)* Assume you are on the surface of a cone with a half angle α which is a surface of revolution about the Z axis. Find an equation in plane polar coordinates for the geodesic curves on this surface. Notice, as in the previous problem, that you can roll up a piece of paper into a cone and visualize these curves geometrically. Why can't you use the "paper roll" to also answer the question about geodesics on a sphere? What is the essential difference between a cone and cylinder on the one hand and a spherical surface on the other?

Problem 5: *(Variational Principle for quantum mechanics)* The quantum mechanics of a one-dimensional system is described by the Schrödinger equation for the complex wave function $\psi(x, t)$:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi = i\hbar\frac{\partial\psi}{\partial t}, \tag{2.63}$$

where $\hbar$ is Planck's constant $\frac{h}{2\pi}$, m the mass, and $V(x)$ the potential energy. Find a variational principle for quantum mechanics using the two dependent variables ψ, ψ^* (complex conjugate of ψ) and the two independent variables x, t. You can treat ψ, ψ^* as two independent generalized coordinates, since the real and imaginary parts are independent variables. Hint: You will try to make the variation of a double integral of the form below vanish:

$$0 = \delta\iint \mathcal{L}\left(\psi, \psi^*, \frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial t}, \frac{\partial\psi^*}{\partial x}, \frac{\partial\psi^*}{\partial t}, x\right) dx\, dt = 0. \tag{2.64}$$

Furthermore, you can assume that $\mathcal{L}$ is real. It might have pieces of the form $V(x)\psi^*\psi$ or $\frac{\partial\psi^*}{\partial x}\frac{\partial\psi}{\partial x}$, for example. See if you can guess the correct form for $\mathcal{L}$ such that the Euler–Lagrange equations lead to the Schrödinger equation and its complex conjugate. The potential energy $V(x)$ is a real function.

Problem 6*: *(One dependent and three independent variables: an electrostatics problem)*

a) Derive the form of the Euler–Lagrange equation for one dependent variable and three independent variables. You want to use x, y, z as independent variables and a function $\Phi(x, y, z)$ as the single dependent variable. Suppose there is a "known Lagrangian"

$$L(\Phi, \vec{\nabla}\Phi, x, y, z). \tag{2.65}$$

You want to minimize the triple integral

$$I \equiv \iiint_V L\, dx\, dy\, dz, \tag{2.66}$$

where V is the volume of integration. What equation should Φ satisfy to accomplish this? The main difficulty is to know how to do the partial integration we did so easily with one independent variable. There is a useful vector calculus identity you can use to do it:

$$\vec{\nabla} \cdot (\vec{F}G) = \vec{F} \cdot \vec{\nabla}G + G\vec{\nabla} \cdot \vec{F}, \tag{2.67}$$

where $\vec{F}$ and G are arbitrary vector and scalar functions of x, y, z. We will assume the variation of Φ on the boundary of the volume V vanishes, in analogy with the case of one independent variable in the principle of least action. Also, the divergence theorem is useful here:

$$\iiint_V \vec{\nabla} \cdot \vec{F}\, dx\, dy\, dz = \iint_S \vec{F} \cdot d\vec{S}, \tag{2.68}$$

where S is the surface of the volume V. This mathematical result holds for any vector function of x, y, z. Use the divergence theorem to derive the final form of the Euler–Lagrange equations for three independent variables in vector calculus notation.

b) In electrostatics, the energy stored in the electric field is proportional to $\iiint E^2\, dV$, where $\vec{E}$ is the electric field and $dV = dx\, dy\, dz$ is the volume element. (We assume there is no free charge in the volume here.) Show that, if $\vec{E} = -\vec{\nabla}\Phi$, and the stored energy is minimized, while Φ (the electrostatic potential) is held constant on the boundaries, Φ must obey Laplace's equation ($\nabla^2\Phi = 0$). Do this in Cartesian coordinates (x, y, z).

Fermat's Principle

Problem 7: *(Fermat's Principle and the bending of light)* A sugar solution with a nonuniform index of refraction $n[y]$ bends a ray of light passing through the solution, as shown in Figure 2.7. The index $n[y]$ is a decreasing function of y, $y(x)$ is the height of the light ray in the tank, and $0 \leq x \leq D$, the distance along the horizontal. The physical reason why a light wave is bent downwards can be seen by considering wavefronts of the light passing through the medium. Because the light velocity is higher at the top of the wave front than at the bottom, the upper portion travels faster and gets ahead of the lower part. Since the light ray is defined by the normal to the wave front, the light bends downward.

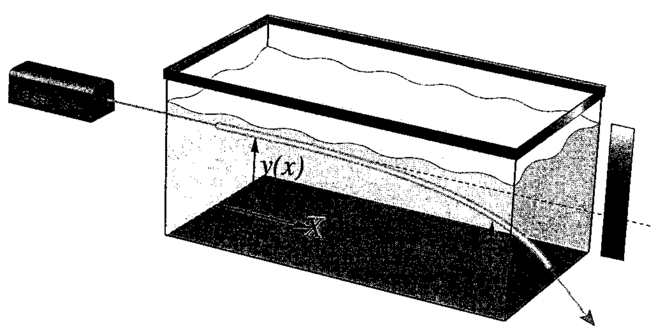

FIGURE 2.7

light beam downwards is observed, and it emerges with $y(30) = -1$ cm. Find the numerical value of α and the approximate shape of the trajectory of the ray in the tank.

Problem 9: *(Brachistochrone 1)* Solve the Brachistochrone problem: Find the function $y(x)$ that connects two fixed points in the XY plane (as shown in Figure 2.1), such that a frictionless mass sliding down the curve arrives at the destination in the least possible time $T[y(x)]$. (Hint: Make the $+x$ direction downwards and the $+y$ direction to the right. This choice is to avoid having to solve for $x(y)$, which you could do instead of turning the axes.) An expression for T, given some trial function $y(x)$, is

$$T = \int dt = \int \frac{ds}{v} = \int \frac{dx}{v}\frac{ds}{dx}, \tag{2.75}$$

where the velocity $v = \frac{ds}{dt}$ and $ds =$ the arc length.

a) Using energy conservation in a constant gravitational field, prove that the form of the functional integral we want is

$$\sqrt{2g}\; T[y] = \int_{x_0}^{x_1} \frac{dx}{\sqrt{x}}\sqrt{1+\left(\frac{dy}{dx}\right)^2}. \tag{2.76}$$

b) Prove that the curve $y(x)$ that minimizes $T[y]$ is

$$y(x) = -\sqrt{x(2r - x)} + 2r \arcsin\left(\frac{x}{2r}\right), \tag{2.77}$$

where "r" is a constant of integration chosen so that the curve passes through the end point. It is really just a scale factor.

c) Plot the curve (using a computer would be helpful here). For $r = 1$, the curve runs between $x = 0$ and $x = 2$. Turn your head 90 degrees to see the Brachistochrone curve!

Problem 10: *(Brachistochrone 2)*

a) Derive the differential equation for $y(x)$ by minimizing the expression for the time using Equation (2.75).

b) Now assume that there is a parameter θ and that, in terms of θ, $x(\theta) = a(1 - \cos\theta)$. (We are still using rotated coordinates, with $+x$ vertically downward, so a is a negative constant.) Using the equation you have obtained for minimizing the time, prove that

$$y(\theta) = |a|(\theta - \sin\theta) \tag{2.78}$$

is a solution, assuming that the particle starts from the origin $\theta = 0$.

c) Graph this curve, which is a cycloid curve. Explain why it is the curve traced out by a point on a wheel of radius $|a|$ rolling down the $+Y$ axis.

d) Calculate the time taken to slide down this curve, assuming that θ varies from 0 to $\frac{\pi}{2}$. Compare it to the time taken to slide down a straight line from the origin to this end point.

Problem 11: *(Ski race)* Imagine you are standing on top of a mountain. The altitude is given by the z coordinate. The shape of the surrounding hills is given by $z = f(x, y)$, where f is a known function. You are an Olympic skier in a race to get to the finish line located down in the valley at a point x_1, y_1.

a) What route should you choose to win the race? First find a set of differential equations for $x(t)$, $y(t)$, then explain how you would find the solution you want.

b) Solve the equations if $f(x, y)$ is the function

$$z = f(x, y) = (\sin^2 2\pi x)(\sin^2 2\pi y). \tag{2.79}$$

Start at $x = y = 0.25$ and ski down to $x = y = 0$. First make a 3-D plot of the surrounding hills with a computer and guess which route you should take. (Hint: Notice that this problem is symmetric under the exchange of x and y.)

Problem 12*: *(Snell's Law)* An open question for physics up to the start of the nineteenth century was about the nature of light: Does light consist of particles or waves? By observing the refraction of light at the interface between two media (say vacuum and glass), and measuring the speed of light in both media, it would have been possible to decide this question.

a) First assume that light consists of a stream of classical particles and that the light is bent towards the perpendicular to the interface as it passes from the vacuum into the glass. $v_{\text{vacuum}} = v_1$, $v_{\text{glass}} = v_2$ as in Figure 2.8. Could there be a transverse force (along the interface plane) exerted on the particles? If not, how is the change in direction related to the relative speed in the two different media? Derive an equation of the form

$$\frac{\sin\theta'}{\sin\theta} = f\left(\frac{v_2}{v_1}\right). \tag{2.80}$$

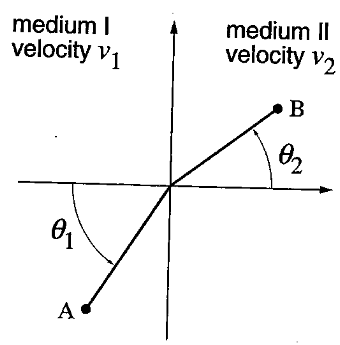

FIGURE 2.8

$$V = \frac{1}{2}\tau\int_0^L dx\left(\frac{\partial y}{\partial x}\right)^2. \tag{2.84}$$

Find the equation of motion for the string. Notice that it has traveling wave solutions of the form $y(x,t) = f(x \pm ct)$, with f an arbitrary function. Find the wave velocity c.

Lagrange Multipliers

Problem 14*: *(Rolling hoop)* A hoop of mass M and radius R rolls without slipping down an inclined plane which makes an angle α with the horizontal. Gravity acts on the hoop in the vertical direction. You can assume that the potential energy of the hoop is the same as if all of its mass were concentrated at the center of the hoop. Using Lagrangian mechanics, find the equation of motion of the hoop.

This problem can be done in at least two different ways. Since there is only one degree of freedom, you can choose the angle ϕ through which the hoop has rolled and write the Lagrangian only in these terms. Or else you can use the distance along the hypotenuse of the plane d as well as ϕ plus a Lagrange multiplier that expresses the rolling constraint $d = R\phi$. In this example, the rolling constraint is holonomic because only one-dimensional motion is involved.

Problem 15: *(Rolling penny on an inclined plane)* Set up the Lagrangian for the problem of the penny on the inclined table (2.57). First calculate the kinetic energy (2.55) for rolling ($\dot\phi \neq 0$) and spinning ($\dot\theta \neq 0$). (Hint: Work out the kinetic energy as a function of $\dot\phi, \dot\theta$ for a ring of radius r; then integrate to get the kinetic energy for a uniform disk.) Then set up the Lagrangian and the constraint equations.

Problem 16*: *(Maximizing the area under a string of fixed length)* This problem involves an elementary application of the method of Lagrange multipliers. A string of fixed length l is placed with its ends on the X axis at $x = \pm a$ as shown in Figure 2.10. The problem is to find the curve $y(x)$ that maximizes the area between the curve and

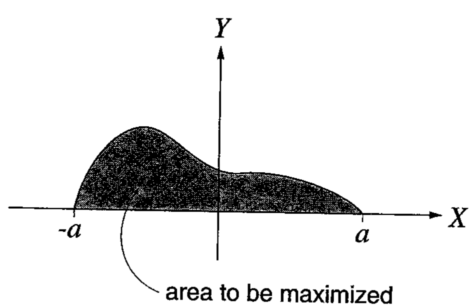

FIGURE 2.10

the X axis:

$$\mathcal{A} = \int_{-a}^{a} y\, dx. \tag{2.85}$$

The intuitive answer is fairly obvious. What is it? The length of the string is given by

$$l = \int_{-a}^{a} \sqrt{1 + y'^2}\, dx, \tag{2.86}$$

where l is fixed (i.e., constrained) and $y' \equiv \frac{dy}{dx}$.

a) Since arbitrary variations $\delta y(x)$ are not possible (why not?), you can't use the calculus of variations directly. But there is a way to do the problem using Lagrange multipliers. Consider introducing an arbitrary constant λ and then maximizing the functional

$$K[y] \equiv \mathcal{A} + \lambda l. \tag{2.87}$$

If, for arbitrary variations $\delta y(x)$, you have $\delta K = 0$, then for the special variations δy that leave the string's length unchanged ($\delta l = 0$), it will be true that $\delta \mathcal{A} = 0$. (Make sure you understand the logic of this last statement.) Find the differential equation from the variational derivative:

$$\frac{\delta}{\delta y}\left[y + \lambda\sqrt{1 + y'^2}\right] = 0. \tag{2.88}$$

b) Integrate this equation once to find $y'(x)$ explicitly. Choose the integration constant so that $y'(0) = 0$. (Symmetry implies $y(x)$ is an even function of x.)

c) Integrate a second time to find the most general form of $y(x)$.

d) Evaluate the up-to-now unknown constant λ as a function of a and l. You may want to use the mathematical integral

$$\int_0^{\alpha} \frac{du}{\sqrt{1 - u^2}} = \arcsin \alpha. \tag{2.89}$$

Did this solution agree with your intuition?

Problem 17: *(Particle in a constant magnetic field)* Inside a solenoid it is a good approximation to regard the magnetic field as constant and directed along the Z axis. Particle motion in such a field is a helical orbit, with particles that start from the axis eventually returning to the axis. Since in a magnetic field the kinetic energy must be constant, this would lead erroneously to the conclusion that Maupertuis' Principle would mean that $\delta \int ds = 0$. Explain why this is not true. Minimizing the arc length gives orbits that are straight lines instead of helices. What is wrong with this argument?

P31B, HW #3 SOLUTIONS DUE: WED, 13 FEB

CORRECTIONS? pt267@cornell.edu

1. (H&F 2.6) AN ELECTROSTATICS PROBLEM

AS SUGGESTED IN THE PROBLEM, WRITE Φ AS THE DEPENDENT VARIABLE & (x,y,z) AS INDEPENDENT VARIABLES.

a) GIVEN SOME $L[\Phi, \vec{\nabla}\Phi, x, y, z]$, WANT EXTREMA OF $I = \int d^3x\, L$. WE NOW FOLLOW THE SAME STEPS AS §2.2 OF H&F, BUT NOW WITH MANY INDEP. VARS.

$$\delta I = I[\Phi + \delta\Phi] - I[\Phi]$$
$$= \int d^3x \left\{ L[\Phi + \delta\Phi, \nabla\Phi + \delta\nabla\Phi, x, y, z] - L[\Phi, \nabla\Phi, x, y, z] \right\}$$

TAYLOR EXPAND:

$$L[\Phi + \delta\Phi, \nabla\Phi + \delta\nabla\Phi] = L[\Phi, \nabla\Phi] + \frac{\partial L}{\partial \Phi}\delta\Phi + \frac{\partial L}{\partial \vec{\nabla}\Phi}\cdot\delta\vec{\nabla}\Phi$$

IT SHOULD BE CLEAR THAT THIS MEANS:

$$\frac{\partial L}{\partial\left(\frac{\partial\Phi}{\partial x}\right)}\delta\left(\frac{\partial\Phi}{\partial x}\right) + \frac{\partial L}{\partial\left(\frac{\partial\Phi}{\partial y}\right)}\delta\left(\frac{\partial\Phi}{\partial y}\right) + \frac{\partial L}{\partial\left(\frac{\partial\Phi}{\partial z}\right)}\delta\left(\frac{\partial\Phi}{\partial z}\right)$$

$$\rightarrow \delta I = \int d^3x \left[\frac{\partial L}{\partial \Phi}\delta\Phi + \underbrace{\frac{\partial L}{\partial \vec{\nabla}\Phi}\cdot\delta\vec{\nabla}\Phi}_{\text{integrate by parts}} \right]$$

USE THE MULTIVARIABLE CHAIN RULE: $\vec{\nabla}\cdot(\vec{F}G) = \vec{F}\cdot\vec{\nabla}G + G\vec{\nabla}\cdot\vec{F}$

$$\Rightarrow \frac{\partial L}{\partial \vec{\nabla}\phi}\cdot\underbrace{\vec{\nabla}(\delta\phi)}_{\delta\vec{\nabla}\phi = \vec{\nabla}\delta\phi} = \vec{\nabla}\cdot\left(\frac{\partial L}{\partial\vec{\nabla}\phi}\delta\phi\right) - \delta\phi\,\vec{\nabla}\cdot\frac{\partial L}{\partial\vec{\nabla}\phi}$$

NOW PERFORM THE d^3x INTEGRAL, INVOKE STOKES' THM:

$$\underbrace{\int d^3x\, \vec{\nabla}\cdot\vec{F}}_{\text{OVER VOLUME}} = \underbrace{\oint \vec{F}\cdot d\vec{s}}_{\text{OVER SURFACE BOUNDING THE VOLUME}}$$

$$\int d^3x \frac{\partial L}{\partial\vec{\nabla}\phi}\cdot\nabla(\delta\phi) = \underbrace{\oint \delta\phi \frac{\partial L}{\partial\vec{\nabla}\phi}\cdot d\vec{s}}_{\substack{\text{BY ASSUMPTION} \\ \delta\phi|_{\text{BOUNDARY}}=0 \\ \text{SO THIS TERM VANISHES}}} - \int d^3x\, \delta\phi\,\vec{\nabla}\cdot\frac{\partial L}{\partial\vec{\nabla}\phi}$$

PLUGGING BACK INTO THE EXPRESSION FOR δI:

$$\delta I = \int d^3x \left[\frac{\partial L}{\partial\phi}\delta\phi - \delta\phi\,\vec{\nabla}\cdot\frac{\partial L}{\partial\vec{\nabla}\phi}\right] = 0 \quad \forall\, \delta\phi$$

SINCE $\delta\phi$ IS ARBITRARY, $\delta I = 0$ IMPLIES:

$$\boxed{\frac{\partial L}{\partial\phi} - \vec{\nabla}\cdot\frac{\partial L}{\partial\vec{\nabla}\phi} = 0}$$

[PROBLEM 1, CONTINUED]

b) WE WANT TO MINIMIZE $\int d^3x \, (\vec{\nabla}\phi)^2$ SUBJECT TO $\phi|_{\text{BOUNDARY}}$ = CONSTANT. USE $L = (\vec{\nabla}\phi)^2$ IN PREVIOUS PART.

$$\underbrace{\frac{\partial L}{\partial \phi}}_{=0} - \vec{\nabla} \cdot \underbrace{\frac{\partial L}{\partial \vec{\nabla}\phi}}_{= 2\vec{\nabla}\phi} = -2\nabla^2 \phi = 0$$

$$\Rightarrow \boxed{\nabla^2 \phi = 0}$$ ← LAPLACE EQ.

2. [H&F #9] BRACHISTOCHRONE 1

TIME FOR A PATH $y(x)$: $T[y] = \int \frac{ds}{v} = \int \frac{dx}{v} \frac{ds}{dx}$

a) WE'VE PICKED COORDINATES WHERE THE INITIAL HEIGHT IS $x = 0$. THE ENERGY IS $E = \frac{1}{2}mv^2 - mgx = 0$

$\Rightarrow v = \sqrt{2gx}$ . .

FURTHER, THE INFINITESIMAL ARCLENGTH IS $ds = dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$.

PLUGGING INTO $T[y]$ ABOVE:

$$\boxed{T[y] = \int \frac{dx}{\sqrt{2gx}} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}}$$

(PROBLEM #2, CONT'D)

b) $L[y,y',x] = \left(\frac{1+(y')^2}{x}\right)^{1/2}$

OBSERVE : $\frac{\partial L}{\partial y} = 0 \Rightarrow \frac{d}{dx}\frac{\partial L}{\partial y'} = 0$

$\Rightarrow P_y = \partial L/\partial y'$ CONSERVED

$$\frac{\partial L}{\partial y'} = \frac{1}{\sqrt{x}}\frac{y'}{\sqrt{1+(y')^2}} \rightarrow P_y^2 = \frac{(y')^2}{x(1+(y')^2)}$$

MASSAGE INTO $y' = f(x)$:

$$P_y^2 x + P_y^2 x (y')^2 = (y')^2$$

$$\Rightarrow (y')^2 = \frac{P_y^2 x}{1-P_y^2 x} = \frac{x^2}{\frac{x}{P_y^2} - x^2}$$

$$\Rightarrow y = \int \frac{x\,dx}{\sqrt{\frac{1}{P_y^2}x - x^2}}$$

IDEA: MASSAGE INTO $\int \frac{dx}{\sqrt{a^2-x^2}}$

WHICH WE CAN SOLVE VIA TRIG SUBSTITUTION,

$x = a\sin\theta$

DEFINE: $\tilde{x} = x - \frac{1}{2P_y}$ & COMPLETE THE SQUARE:

$$\Rightarrow y = \int \frac{\tilde{x} + \frac{1}{2P_y}}{\sqrt{1/4P_y^2 - \tilde{x}^2}}\,d\tilde{x}$$

$$= \int_{-1/2P_y}^{x-\frac{1}{2P_y}} \frac{\tilde{x}\,d\tilde{x}}{\sqrt{\frac{1}{4P_y^2} - \tilde{x}^2}} + \frac{1}{2P_y}\int_{-1/2P_y}^{x-\frac{1}{2P_y}} \frac{d\tilde{x}}{\sqrt{\frac{1}{4P_y^2} - \tilde{x}^2}}$$

FOR SIMPLICITY (& WITH SOME FORESIGHT), DEFINE

$$r = \frac{1}{2P_y} \quad \text{s.t.} \quad \tilde{x} = x - r$$

$$y = \underbrace{\int_{-r}^{x-r} \frac{\tilde{x}\, d\tilde{x}}{\sqrt{r^2 - \tilde{x}^2}}}_{\text{I}} + \underbrace{r \int_{-r}^{x-r} \frac{d\tilde{x}}{\sqrt{r^2 - \tilde{x}^2}}}_{\text{II}}$$

I. ALGEBRAIC SUBSTITUTION : $u = r^2 - \tilde{x}^2$, $d\tilde{x} = -\frac{du}{2\tilde{x}}$

$$\text{I} = \int_0^{2xr - x^2} \frac{-1}{2} \frac{du}{\sqrt{u}} = -\sqrt{u}\,\Big|_0^{x(2r-x)} = -\sqrt{x(2r-x)}$$

II. TRIGONOMETRIC SUBSTITUTION: $\tilde{x} = r\sin\theta$, $d\tilde{x} = r\cos\theta\, d\theta$

$$\text{II} = r\int_{\sin^{-1}(-1)}^{\sin^{-1}\left(\frac{x-r}{r}\right)} \frac{\cancel{r\cos\theta}\, d\theta}{\cancel{\sqrt{r^2\cos^2\theta}}} = r\left[\sin^{-1}\left(\frac{x-r}{r}\right) + \frac{\pi}{2}\right]$$

$$\Rightarrow \boxed{y(x) = -\sqrt{x(2r-x)} + \underbrace{r\left(\sin^{-1}\left(\frac{x-r}{r}\right) + \frac{\pi}{2}\right)}}$$

remark : $= r\cos^{-1}\left(1 - \frac{x}{r}\right)$

[pf: DRAW A PICTURE]

REMARK 2: $r\cos^{-1}\left(1 - \frac{x}{r}\right) = 2\sin^{-1}\left(\sqrt{\frac{x}{2r}}\right)$

... LOOKS LIKE THE BOOK HAS A TYPO – THERE SHOULD BE A SQUARE ROOT IN THE ARGUMENT!

Problem 2, H&F 2.9: Brachistochrone

```
Plot[
 -√(x (2 r - x)) + 2 r ArcSin[√(x/(2 r))] /. {r → 1},
 {x, 0, 2},
 PlotStyle → {Red, Thick},
PlotRange → {{0, 2}, {0, 2}},
FrameLabel → {"x (depth)", "y[x] (horizontal position)"},
Frame → True,
 PlotLabel → "Brachistochrone",
 Filling → None]
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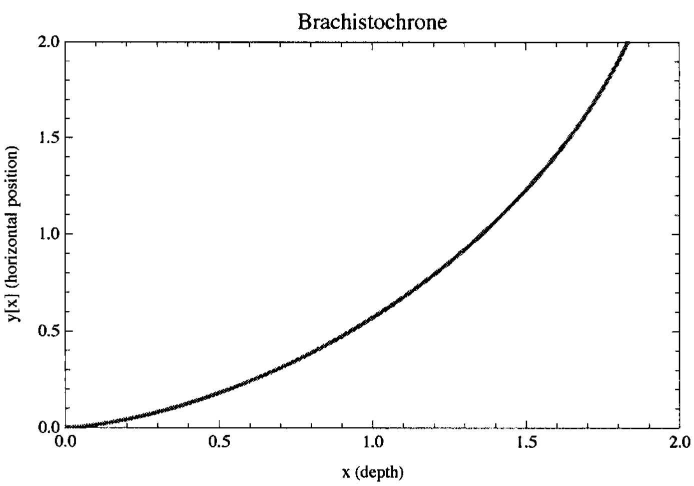


3. [H&F 2.10] BRACHISTOCHRONE 2

a) WRITE OUT EOM FOR $y(x)$ FROM PVS. PROBLEM :

$$0 = \frac{d}{dx}\left(\frac{1}{\sqrt{x}}\frac{y'}{\sqrt{1+(y')^2}}\right)$$

$$= -\frac{1}{2}\frac{x^{-3/2}\,y'}{\sqrt{1+(y')^2}} + \frac{x^{-1/2}\,y''}{\sqrt{1+(y')^2}} - \frac{x^{-1/2}(y')^2 y''}{(1+(y')^2)^{3/2}}$$

$$= \boxed{\frac{x^{-1/2}\,y'}{\sqrt{1+(y')^2}}\left(-\frac{1}{2x} + \frac{y''}{y'} - \frac{y'\,y''}{1+(y')^2}\right)}$$

FOR PART (b), IT WILL BE USEFUL TO WRITE THIS w/ NO FRACTIONS:

$$(*) \qquad \boxed{0 = \frac{1}{x^{3/2}(1-(y')^2)^{3/2}}\left(-\frac{1}{2}y'\left[1+(y')^2\right] + x y''\left[1+(y')^2\right] - x(y')^2 y''\right)}$$

b) TO AVOID THE ABS. VALUE, DEFINE $A \equiv -a > 0$.

$$x(\theta) = -A(1-\cos\theta) \rightarrow \partial x/\partial\theta = -A\sin\theta$$

$$y(\theta) = A(\theta - \sin\theta) \rightarrow \partial y/\partial\theta = A(1-\cos\theta)$$

WE WANT TO SHOW THAT THIS SATISFIES (*), BUT WE HAVE TO BE CAREFUL! y' IN (*) MEANS $\partial y/\partial x$, NOT $\partial y/\partial\theta$.

$$\rightarrow \quad y' = \frac{\partial y}{\partial x} = \frac{\partial y}{\partial\theta}\cdot\frac{\partial\theta}{\partial x} = \frac{\partial y}{\partial\theta}\left(\frac{\partial x}{\partial\theta}\right)^{-1}$$

$$y'' = \frac{\partial}{\partial x}\frac{\partial y}{\partial x} = \frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta}\frac{\partial y}{\partial x}$$

$$\frac{\partial y}{\partial x} = \frac{A(1-\cos\theta)}{-A\sin\theta} = \frac{-(1-\cos\theta)}{\sin\theta}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{-A\sin\theta}\underbrace{\frac{\partial}{\partial\theta}\left(\csc\theta-\cot\theta\right)}_{(\csc\theta\cot\theta-\csc^2\theta)} = \frac{1-\cos\theta}{A\sin^3\theta}$$

IT IS ALSO USEFUL TO WRITE OUT:

$$1+\left(\frac{\partial y}{\partial x}\right)^2 = \frac{\sin^2\theta+(\cos^2\theta-2\cos\theta+1)}{\sin^2\theta} = 2\,\frac{1-\cos\theta}{\sin^2\theta}$$

NOW LET'S SHOW THAT (*) VANISHES. WRITE $c=\cos\theta$, $s=\sin\theta$. FURTHER, FOCUS ONLY ON THE SUM OF TERMS IN PARENTHESES:

$$\underbrace{-\frac{1}{2}y'[1+(y')^2]}_{\mathrm{I}} + \underbrace{xy''[1+(y')^2]}_{\mathrm{II}} - \underbrace{x(y')^2y''}_{\mathrm{III}}$$

$$\mathrm{I} = -\frac{1}{2}\cdot\frac{-(1-c)}{s}\cdot 2\,\frac{1-c}{s^2} = \frac{1}{s^3}(1-c)^2$$

$$\mathrm{II} = -A(1-c)\cdot\frac{1-c}{As^3}\cdot 2\,\frac{1-c}{s^2} = -\frac{2}{s^5}(1-c)^3$$

$$\mathrm{III} = +A(1-c)\cdot\frac{(1-c)^2}{s^2}\cdot\frac{1-c}{As^3} = +\frac{1}{s^5}(1-c)^4$$

$$\mathrm{I}+\mathrm{II}+\mathrm{III} = \frac{(1-c)^2}{s^5}\left(s^2-2(1-c)+(1-c)^2\right)$$

$$\mathrm{I}+\mathrm{II}+\mathrm{III} = \frac{(1-c)^2}{s^5}\left(s^2-2+2c+1-2c+c^2\right) = \frac{(1-c^2)}{s^5}\left(1-2+2c+1-2c\right)$$

$$= \boxed{0}$$ ✓

It's a miracle!

Remark: there's a much easier way to do this...

FROM PART (a): EOM is $\underbrace{\frac{d}{dx}}_{\left(\frac{\partial x}{\partial \theta}\right)^{-1}\frac{\partial}{\partial \theta}}\left(\frac{x^{-1/2}\, y'}{\sqrt{1+(y')^2}}\right) = 0$

(SO IF THIS IS θ - INDEP., THEN THE EOM IS SATISFIED

RECALL: $x = -A(1-\cos\theta)$

$y' = -(1-\cos\theta)/\sin\theta$

$1+(y')^2 = 2(1-\cos\theta)/\sin^2\theta$

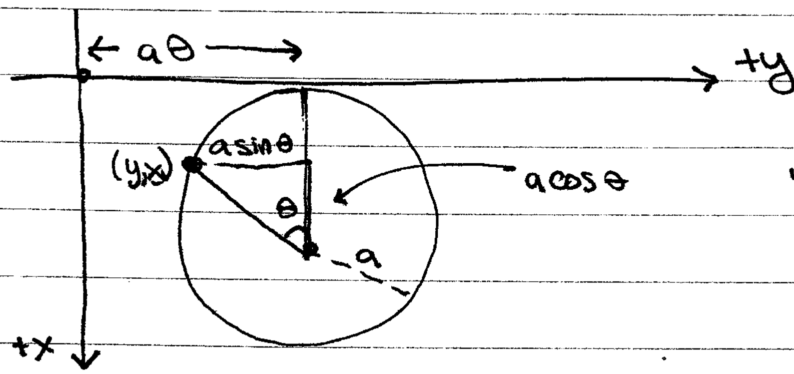
$$\Rightarrow \frac{x^{-1/2}\, y'}{\sqrt{1+(y')^2}} = \frac{1}{\sqrt{-A(1-\cos\theta)}}\,\frac{-(1-\cos\theta)}{\sin\theta}\,\frac{\sin\theta}{\sqrt{2(1-\cos\theta)}} = \frac{-1}{\sqrt{-2A}}$$

this is indeed independent of $\theta \Rightarrow$ $\boxed{\text{EOM SATISFIED}}$ ✓

c) SEE ATTACHED Mathematica PLOT

c) [continued]

THIS CORRESPONDS TO THE CURVE TRACED BY A POINT ON A WHEEL ROLLING DOWN THE y-AXIS:



"proof by picture"

$$\Rightarrow \quad x(\theta) = a(1-\cos\theta) \qquad dx = a\sin\theta \, d\theta$$

$$y(\theta) = a(\theta - \sin\theta) \qquad dy = a(1-\cos\theta)\, d\theta$$

d) $$T[y] = \frac{1}{\sqrt{2g}} \int_{\theta=0}^{\theta=\pi/2} \frac{1}{\sqrt{x}} \sqrt{dx^2 + dy^2}$$

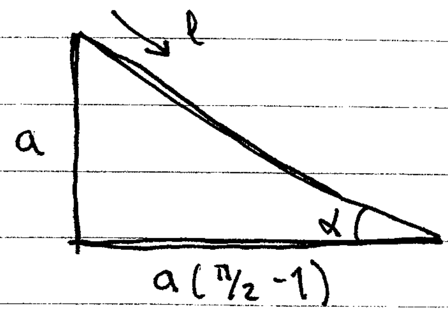
$$= \frac{1}{\sqrt{2g}} \int_{\theta=0}^{\theta=\pi/2} \frac{a}{\sqrt{a(1-\cos\theta)}} \sqrt{\sin^2\theta + 1 - 2\cos\theta + \cos^2\theta}\, d\theta$$

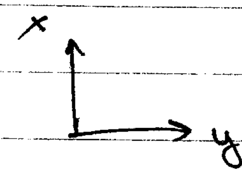
$$= \frac{1}{\sqrt{2g}} \int_{\theta=0}^{\theta=\pi/2} \frac{\sqrt{a}}{\sqrt{g}} \sqrt{\frac{\cancel{1-\cos\theta}}{\cancel{1-\cos\theta}}}\, d\theta$$

$$= \boxed{\sqrt{\frac{a}{g}} \cdot \frac{\pi}{2}}$$

(d), CONTINUED. COMPARE TO STRAIGHT LINE PATH:

$$\left.\begin{array}{ll} x_0 = 0 & x(\pi/2) = a \\ y_0 = 0 & y(\pi/2) = a(\pi/2 - 1) \end{array}\right\} \text{ signs don't matter}$$





YOU CAN PLUG INTO THE PREVIOUS EQUATION, OR JUST USE, FRESHMAN PHYSICS: FORCE ALONG INCLINE: $F_\ell = mg \sin\alpha$

$$\Rightarrow \ddot{\ell} = g \sin\alpha$$

$$\ell = \tfrac{1}{2} g \sin\alpha \, t^2 \qquad \sin\alpha = a/\ell$$

$$T_{str} = \sqrt{2\ell / g\sin\alpha}$$

$$= \sqrt{\frac{2}{ag}}\,\ell \quad \longleftarrow \quad \ell = a\sqrt{1 + (\pi/2 - 1)^2}$$

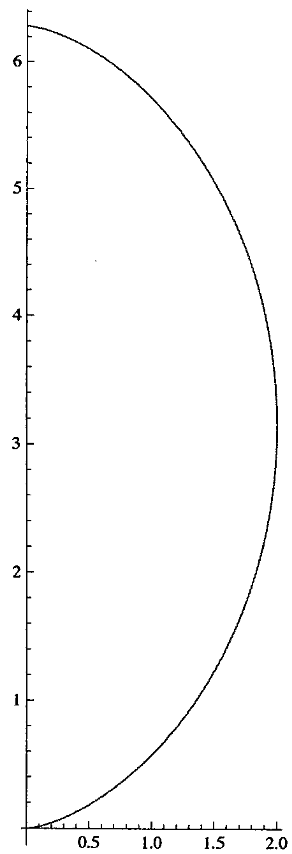
$$= \sqrt{\frac{a}{g}}\;\underbrace{\sqrt{2 + 2(\tfrac{\pi}{2} - 1)^2}}$$

$$= 1.63 \quad > \quad \frac{\pi}{2} = 1.57$$

$$\Rightarrow \boxed{T(\text{straight line}) > T(\text{cycloid})}$$

Problem 3, H&F 2.10: Brachistocrone II

```
x[t_] := a (1 - Cos[t])
y[t_] := Abs[a] (t - Sin[t])
ParametricPlot[{x[t], y[t]} /. {a → 1}, {t, 0, 2 Pi}]
```



Here's a nice animation to see the motion of a point. (Try typing this into *Mathematica*)

```
Animate[
 ListPlot[{{x[t], y[t]} /. {a → 1}},
  PlotRange → {{0, 2 π}, {0, 2 π}},
  PlotStyle → PointSize[Large]
 ],
 {t, 0, 2 π},
 AnimationRunning → False
]
```

4. (H&F 2.16) AREA UNDER A STRING (fixed len)

INTUITION: this better be a semi circle!

a) ARBITRARY VARIATIONS ARE NOT ALLOWED BECAUSE THEY WOULD NOT SATISFY THE FIXED STRING LENGTH CONSTRAINT.

$$K[y] = \int_{-a}^{a} \{ \underbrace{y + \lambda\sqrt{1+y'^2}}_{L} \} dx$$

y_{EOM}: $\frac{d}{dx}\frac{\partial L}{\partial x'} - \frac{\partial L}{\partial y} = 0$

$$\Rightarrow \boxed{0 = \frac{d}{dx}\left(\frac{\lambda y'}{\sqrt{1+y'^2}}\right) - 1}$$

b) $\frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = \frac{1}{\lambda}$

$$\rightarrow \frac{y'}{\sqrt{1+y'^2}} = \frac{x}{\lambda}$$ USING $y'(0)=0$

$$\rightarrow \lambda^2 (y')^2 = (y')^2 x^2 + x^2$$
$$(y')^2(\lambda^2 - x^2) = x^2$$

choose + solution of square root, will correspond to upper or lower semicircle.

$$\boxed{y'(x) = \sqrt{\frac{x^2}{\lambda^2 - x^2}}}$$

c) $$y(x) = \int_{-a}^{x} \frac{\tilde{x}\, d\tilde{x}}{\sqrt{\lambda^2 - \tilde{x}^2}}$$ we did this integral via u-subst. in problem 2. $u = \lambda^2 - \tilde{x}^2$

$$= \int_{\lambda^2-a^2}^{\lambda^2-x^2} \frac{-du}{2\sqrt{u}}$$

$$= -\sqrt{u} \Big|_{\lambda^2-a^2}^{\lambda^2-x^2}$$

integration constant s.t. $y(\pm a) = 0$

$$\boxed{y(x) = -\sqrt{\lambda^2 - x^2} + \sqrt{\lambda^2 - a^2}}$$

d) PLUG INTO $l = \int_{-a}^{a} \sqrt{1 + y'^2}\, dx$

$$1 + (y')^2 = 1 + \frac{x^2}{\lambda^2 - x^2} = \frac{\lambda^2}{\lambda^2 - x^2}$$

$$\Rightarrow \frac{l}{2} = \int_0^a \frac{\lambda}{\sqrt{\lambda^2 - x^2}}\, dx$$ another integral we did in problem 2

$$= \lambda \int_0^{a/\lambda} \frac{1}{\sqrt{1 - (x/\lambda)^2}}\, d\left(\frac{x}{\lambda}\right)$$

$$= \lambda \sin^{-1}\left(\frac{a}{\lambda}\right)$$

$$\Rightarrow \boxed{\frac{a}{\lambda} = \sin\left(\frac{l}{2\lambda}\right)}$$

CONSIDER THE CASE $l = \pi a$, WHERE WE REALLY EXPECT THE SOLUTION TO BE A SEMICIRCLE. IT'S CLEAR THAT $\lambda = a$ SATISFIES THE ABOVE EQUATION, AND THE EXPRESSION FOR $y(x)$ IS INDEED THAT OF A SEMICIRCLE. ✓

5. HARMONIC OSCILLATOR w/ HARMONIC VARIATIONS

(a) $$\delta S = \int_0^T \left[\tfrac{1}{2} m \left(\dot{\tilde{x}} + \delta\dot{x}\right)^2 - \tfrac{1}{2} m\omega_0^2 \left(\tilde{x} + \delta x\right)^2\right] dt - S[x]$$

$$= \int_0^T \Big[\underbrace{m\dot{\tilde{x}}\,\delta\dot{x} - m\omega_0^2 \tilde{x}\,\delta x}_{\text{THIS BETTER VANISH}} \Big] dt + \underbrace{S[\delta x]}_{\text{what we want}}$$

$$= m\Big[\underbrace{\ddot{\tilde{x}} - \omega_0^2 \tilde{x}}_{=0 \text{ BY EQUATION of MOTION FOR } \tilde{x}} \Big]\delta x$$ integrate by pts. & BOUNDARY COND.

$$= \boxed{S[\delta x]} \quad \checkmark$$

(b) $$\delta S = \int_0^T \left[\frac{m}{2}\left(\frac{N\pi\Delta}{T}\cos\left(\frac{N\pi t}{T}\right)\right)^2 - \frac{m\omega_0^2}{2}\Delta^2 \sin^2\frac{N\pi t}{t} \right] dt$$

Use: $\int_0^T \cos^2\left(\frac{N\pi t}{T}\right) dt = \int_0^T \sin^2\left(\frac{N\pi t}{T}\right) dt = \frac{T}{2}$ for $N \in \mathbb{Z}$

$$= \frac{m}{2}\left(\frac{N\pi\Delta}{T}\right)^2 \frac{T}{2} - \frac{m}{2}\omega_0^2 \Delta^2 \frac{T}{2}$$

$$= \frac{m\Delta^2}{4T}\left(N^2\pi^2 - \omega_0^2 T^2\right) \Rightarrow \boxed{c_N = \frac{m}{4T}\left(N^2\pi^2 - \omega_0^2 T^2\right)}$$

We can see that $c_N > 0$ if $\omega_0 T < N\pi$

< 0 if $\omega_0 T > N\pi$