$\boxed{\text{SOLUTION SET}}$

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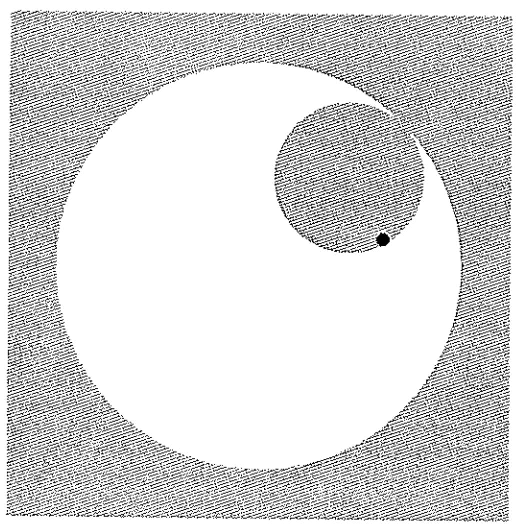
Assignment 1

Due date: Wednesday, January 30

1. H&F 1.2
2. H&F 1.3
3. H&F 1.4
4. H&F 1.6
5. H&F 1.8
6. A massless wheel of radius r rolls without slipping around the circumference of a circular hole of radius R as shown below:



The only mass in this system is a point mass m fixed on the rim of the wheel.

(a) Neglecting gravity (the wheel moves in a horizontal plane) express the kinetic energy for this system in terms of the angle $\theta(t)$ that describes the position of the wheel's center about the center of the hole. Assume the mass is attached so that it touches the hole when $\theta = 0$.

(b) Using the fact you learned from elementary mechanics, that mechanical energy is conserved (in the absence of friction), determine how the angular velocity $\dot{\theta}$ depends on θ.

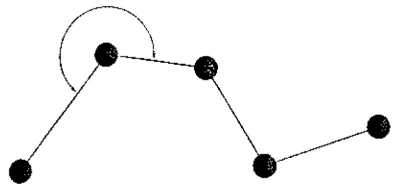


FIGURE 1.6

reasoning at every step. It is a good habit to get into. The object is not only to solve the problem, but also to be able to explain it. The extra time invested is worth it!

Degrees of Freedom

Problem 1: *(Bicycle)* Make a simplified model of a bicycle. How many degrees of freedom are there? Restrict your model to the most important degrees of freedom.

Problem 2: *(Flexible chain)* A flexible chain of M massive point particles has rigid weightless rods as $M-1$ links as shown in Figure 1.6. Each joint is free to move in any direction. How many degrees of freedom does the chain have? If you place the chain on a flat table, how many degrees of freedom does it then have? Finally, suppose the chain is lifted off the table and is closed by one more link. How many degrees of freedom are there then?

Dot Cancellation

Problem 3: *(Spherical polar coordinates)* Prove that the relation $\frac{\partial \vec{r}_i}{\partial q_k} = \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_k}$ (Equation (1.44)) holds if you have a one particle system described by spherical polar coordinates: Choose for q_1, q_2, q_3 the parameters r, θ, ϕ.

Kinetic Energy

Problem 4: *(Spherical pendulum)* Consider the spherical pendulum, which consists of a mass m suspended by a string from the ceiling. The mass is free to swing in both directions but maintains a constant distance from the point of suspension. Choose spherical polar coordinates θ, ϕ as generalized coordinates for this problem. What is $T(\theta, \dot{\theta}, \phi, \dot{\phi})$?

Virtual Work

Problem 5: *(Spring pendulum)* Imagine that you have a pendulum made of a mass hanging from a spring. Unlike the previous problem, restrict all motion to take place in a vertical plane here. At rest the pendulum has a length l_0. The spring constant is k. There are two degrees of freedom, which you can take as θ, the angle from the vertical of the pendulum, and x, the extension of the spring. (When the spring is extended, the pendulum length is $l_0 + x$.) Find the generalized forces $\mathcal{F}_\theta$, and $\mathcal{F}_x$ using the principle of virtual work.

Invariance

Problem 6*: † *(Physically equivalent Lagrangians)*

a) Prove that adding a constant to the Lagrangian L or else multiplying the Lagrangian by a constant produces a new Lagrangian L' that is physically equivalent to L. What we mean by physically equivalent is that the Euler–Lagrange equations for the $q(t)$ remain the same (i.e., are invariant) under this change of Lagrangian.

b) There is even more freedom to change the Lagrangian *without changing the physics it describes*. A total time derivative of an arbitrary function of the dynamical variables can be added to the Lagrangian to produce a completely equivalent Lagrangian. Consider a new Lagrangian L' which is produced as follows:

$$L \to L' = L + \frac{dF}{dt}. \tag{1.90}$$

We assume that F is an arbitrary function of the qs and t but is not a function of the $\dot{q}$s. Prove that the Euler–Lagrange equations for $q(t)$ are invariant under this change of Lagrangian. Since one can always make transformations of this sort, the Lagrangian for a given physical system is not unique.

Problem 7*: *(Guessing the Lagrangian for a free particle)* Assume that you do not know about kinetic energy or Newton's Laws of motion. Suppose instead of deriving the Euler–Lagrange equations, we postulated them. We define the basic law of mechanics to be these equations and ask ourselves the question: What is the Lagrangian for a free particle? (This is a particle in empty space with no forces acting on it. Be sure to set up an inertial reference system.)

a) Explain why, on very general grounds, L cannot be a function of x, y, or z. It also cannot depend on the individual coordinates of velocity in any way except as a function of the magnitude of the velocity: $v^2 = v_x^2 + v_y^2 + v_z^2$. On what assumption about the properties of space does this depend?

b) The simplest choice might be to guess it must be proportional to v^2, where $\vec{v}$ is the particle velocity in an inertial frame K. Take $L = v^2$. A second inertial frame K' moves at the constant velocity $-\vec{V}_0$ with respect to K, so that the transformation law of velocities is

$$\vec{v}' = \vec{v} + \vec{V}_0. \tag{1.91}$$

Prove that $L' = v'^2$ is a possible choice for the Lagrangian in the frame K'. Explain how this proves that all inertial frames are equivalent. You will have to

† The symbol "*" will be used to denote problems used in a weekly student seminar that was part of the course taught at Cornell in 1994–1996. In the seminar, student groups had an hour to solve an assigned problem, after which they presented the solution to the class.

make use of the result of the previous problem to show this. With this approach we prove the equivalence of inertial frames from the form of the Lagrangian, instead of postulating this equivalence at the start, which is the usual way of doing things.

c) Instead of proving it, adopt the equivalence of inertial frames as a postulate, in addition to the Euler–Lagrange equations. Explain why this means that

$$L'(v+V_0) = L(v) + \frac{dF(x,t)}{dt}. \tag{1.92}$$

$L(v)$ is an unknown function for the free particle that we are trying to determine from these principles. (Work in one dimension to make things easier.) Let V_0 be an infinitesimal quantity. Expand the left side of Equation (1.92) in a Taylor series and keep only the first two terms. From this prove $L(v) \sim v^2$.

Problem 8: *(Potentials with scaling properties)* Let $V(\vec{r}_1, \ldots, \vec{r}_M)$ be the potential energy of a system of M massive particles which has the scaling property

$$V(\alpha\vec{r}_1, \ldots, \alpha\vec{r}_M) = \alpha^k V(\vec{r}_1, \ldots, \vec{r}_M) \tag{1.93}$$

(k is usually an integer, α an arbitrary constant.) Prove that, if the Lagrangian is to remain invariant (except for multiplication by a constant), and all distances are scaled by a factor α, then the time must be scaled by a factor $\beta = \alpha^{1-\frac{k}{2}}$. Applications of this include:

a) If $k = 1$, the force is constant, like gravity. Prove that distances scale like t^2.

b) If $k = 2$, the force is like that of a harmonic oscillator or a system of harmonic oscillators coupled to each other. Prove that the frequency or frequencies of such a system are independent of the amplitude of oscillation.

c) If $k = -1$, we have the Kepler problem (inverse square force law). Prove Kepler's third law from this scaling law above. (That is, prove $d^3 = t^2$, where d could be any distance in the problem. Normally it is the mean distance of a planet from the sun.)

Hamiltonian Concept/Energy

Problem 9: *(Quadratic forms)* Prove that, if the constraints are scleronomic (i.e., time-independent), T is a *quadratic* function (*quadratic form*) of the generalized velocities. Then prove this implies

$$\sum_k \dot{q}_k \frac{\partial T}{\partial \dot{q}_k} = 2T. \tag{1.94}$$

Assuming that the kinetic energy *is* a quadratic form in the generalized velocities so that the formula above is correct, prove that the Hamiltonian H (Equation (1.65)) is the total energy ($H = T + V = E$).

P318, HW #1 SOLUTIONS DUE: WED, 30 Jan

CORRECTIONS? EMAIL FLIP: pt267@cornell.edu

1. (H&F 1.2) <u>FLEXIBLE CHAIN</u>

- EACH PARTICLE CAN MOVE IN 3 DIRECTIONS.

$$\rightarrow (M \text{ PARTICLES}) \times \left(3 \, \frac{\text{DIRECTIONS}}{\text{PARTICLE}}\right) = 3M \text{ dof}$$

- EACH RIGID ROD RESTRICTS THE MOTION OF ONE PARTICLE RELATIVE TO THE NEXT.

$$\rightarrow ((M-1) \text{ RODS}) \times (1 \text{ CONSTRAINT / ROD}) = -(M-1) \text{ dof}$$

SO THE CHAIN HAS $3M - (M-1) = \boxed{2M+1 \text{ dof}}$

IF WE PUT THE CHAIN ON A TABLE, THE PARTICLES ONLY HAVE 2 dof EACH. THUS WE END UP WITH

$2M - (M-1) = \boxed{M+1 \text{ dof ON THE TABLE}}$

IF WE LIFT THE CHAIN OFF THE TABLE AND CLOSE THE LINK, THEN WE HAVE

$$3M - M = \boxed{2M \text{ dof FOR CLOSED CHAIN}}$$

2. (H & F 1.3) SPHERICAL POLAR COORDINATES

$$x = r\sin\theta\cos\phi \qquad \frac{\partial x}{\partial r} = \sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi \qquad \frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi$$

$$z = r\cos\theta \qquad \frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi$$

$$\dot{x} = \dot{r}\sin\theta\cos\phi + r\cos\theta\cos\phi\,\dot{\theta} - r\sin\theta\sin\phi\,\dot{\phi}$$

$$\frac{\partial \dot{x}}{\partial \dot{r}} = \sin\theta\cos\phi = \frac{\partial x}{\partial r} \qquad \frac{\partial \dot{x}}{\partial \dot{\theta}} = r\cos\theta\cos\phi = \frac{\partial x}{\partial \theta} \qquad \frac{\partial \dot{x}}{\partial \dot{\phi}} = -r\sin\theta\sin\phi = \frac{\partial x}{\partial \phi}$$

& analogously for the y & z components.

[AS EXPECTED FROM THE GENERAL PROOF IN H & F PAGES 15 - 16!]

3. (H&F 1.4) SPHERICAL PENDULUM

IN SPHERICAL COORDINATES:

$$x_1 = r\sin\theta\cos\phi$$
$$x_2 = r\sin\theta\sin\phi$$
$$x_3 = r\cos\theta$$

in this problem, $r = \text{const.}$ (eg $\dot{r} = 0$)

(H&F eq 1.50) $T = \frac{1}{2}m|\dot{\vec{r}}|^2$

$$\dot{x}_1 = r\cos\theta\cos\phi\,\dot{\theta} + r\sin\theta(-\sin\phi)\dot{\phi}$$
$$\dot{x}_2 = r\cos\theta\sin\phi\,\dot{\theta} + r\sin\theta(\cos\phi)\dot{\phi}$$
$$\dot{x}_3 = -r\sin\theta\,\dot{\theta}$$

$c\theta = \cos\theta$, etc

$$T = \frac{1}{2}m\left[\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2\right]$$
$$= \frac{1}{2}m\left[\left(r^2c\theta^2c\phi^2\dot{\theta}^2 \cancel{- 2r^2c\theta s\theta c\phi s\phi\dot{\theta}\dot{\phi}} + r^2s\theta^2s\phi^2\dot{\phi}^2\right) + \left(r^2c\theta^2s\phi^2\dot{\theta}^2 \cancel{+ 2rc\theta s\theta c\phi s\phi\dot{\theta}\dot{\phi}} + r^2s\theta^2c\phi^2\dot{\phi}^2\right) + r^2s\theta^2\dot{\theta}^2\right]$$

$s\phi^2 + c\phi^2 = 1$

$$= \frac{1}{2}mr^2\left[c\theta^2\dot{\theta}^2 + s\theta^2\dot{\phi}^2 + s\theta^2\dot{\theta}^2\right]$$

$$= \boxed{\frac{1}{2}mr^2\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right)}$$

4. (H&F 1.6) PHYSICALLY EQUIVALENT LAGRANGIANS

a) IF $L \to L' = L + c$, THEN THE EULER-LAGRANGE EQUATION DOESN'T CHANGE:

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{q}_k} - \frac{\partial L'}{\partial q_k} = \underbrace{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k}}_{\text{OLD EQ. OF MOTION}} + \underbrace{\left(\frac{d}{dt}\frac{\partial}{\partial \dot{q}_k} - \frac{\partial}{\partial q_k}\right)c}_{\text{DERIVATIVE of } c = 0} \quad ✓$$

SIMILARLY, IF $L \to L' = aL$, THEN

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = 0 \iff \frac{d}{dt}\frac{\partial L'}{\partial \dot{q}_k} - \frac{\partial L'}{\partial q_k} = a\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k}\right) = 0$$

SO THE EULER-LAGRANGE EQ. ALSO DOESN'T CHANGE HERE. ✓

b) NOW TRY $L \to L' = L + \frac{d}{dt}F(\vec{q}(t), t)$

PLUG INTO EULER-LAGRANGE:

$$\underbrace{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k}}_{\text{OLD TERM}} + \underbrace{\frac{d}{dt}\frac{\partial}{\partial \dot{q}_k}\frac{dF}{dt} - \frac{\partial}{\partial q}\frac{dF}{dt}}_{\text{WANT THIS TO VANISH}} \stackrel{?}{=} 0$$

idea: "CANCEL THE DOTS": $\frac{d}{dt}\frac{\partial \dot{F}}{\partial \dot{q}} = \frac{d}{dt}\frac{\partial F}{\partial q}$

USE: $\frac{d}{dt}F = \dot{F} = \frac{\partial F}{\partial q}\dot{q} + \frac{\partial F}{\partial t} \Rightarrow \frac{d}{dt}\frac{\partial}{\partial \dot{q}}\dot{F} = \frac{d}{dt}\frac{\partial F}{\partial q}$

AND: $\frac{\partial}{\partial q}\frac{dF}{dt} = \frac{d}{dt}\frac{\partial F}{\partial q}$

$$\Rightarrow \frac{d}{dt}\frac{\partial}{\partial \dot{q}_k}\frac{dF}{dt} - \frac{\partial}{\partial q}\frac{dF}{dt} = 0,$$

EULER-LAGRANGE EQ FOR L' IS EQUIVALENT TO THAT FOR L. ✓

5. (H&F 1.8) <u>POTENTIALS & SCALING</u>

$$V(\alpha\vec{r}_1, \dots, \alpha\vec{r}_M) = \alpha^k V(\vec{r}_1, \dots, \vec{r}_M)$$

SUPPOSE $L(\alpha\vec{r}_1, \dots, \alpha\vec{r}_M) = \alpha^k L(\vec{r}_1, \dots, \vec{r}_M)$.

↳ from pvs problem this is a physically equivalent problem. why α^k in front? $V(\alpha\vec{r}) \to \alpha^k V(\vec{r})$, THIS SETS THE CONSTANT PREFACTOR FOR $L = T - V$.

THEN: NEED $T(\alpha\vec{r}_1, \dots, \alpha\vec{r}_M) \to \alpha^k T(\vec{r}_1, \dots, \vec{r}_M)$
WITHOUT LOSS of GENERALITY, SUPPOSE $t \to \beta t$
UNDER THIS RESCALING. WHAT DOES $T(\alpha\vec{r}) \to \alpha^k T(\vec{r})$
IMPOSE ON β?

$$T(\alpha\vec{r}_1, \dots, \alpha\vec{r}_M) = \sum_{i=1}^{M} \frac{1}{2} m \left(\frac{d}{d(\beta t)} \alpha\vec{r}\right)^2$$

$$= \left(\frac{\alpha}{\beta}\right)^2 \underbrace{\sum_{i=1}^{M} \frac{1}{2} m \left(\frac{d}{dt}\vec{r}\right)^2}_{T(\vec{r}_1, \dots, \vec{r}_M)}$$

$$\stackrel{?}{=} \alpha^k T$$

REQUIRES $\left(\frac{\alpha}{\beta}\right)^2 = \alpha^k \Rightarrow \boxed{\beta = \alpha^{1-k/2}}$

5a) $\underline{k=1}$: time scales like $\beta = \alpha^{1/2}$

length scales like α

$\Rightarrow$ $\boxed{\text{DISTANCES} \sim \text{TIME}^2}$

b) $\underline{k=2}$: time $\sim \alpha^0$

FREQUENCY $\sim 1/t \sim \alpha^0$

AMPLITUDE $\sim |\vec{r}| \sim \alpha$

$\Rightarrow$ $\boxed{\text{FREQ. INDEP OF AMPLITUDE}}$

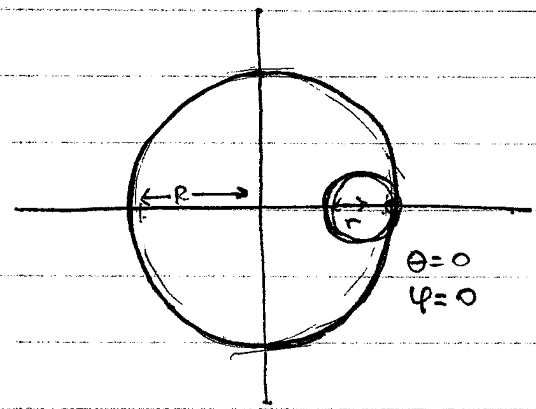
otherwise scaling doesn't make sense

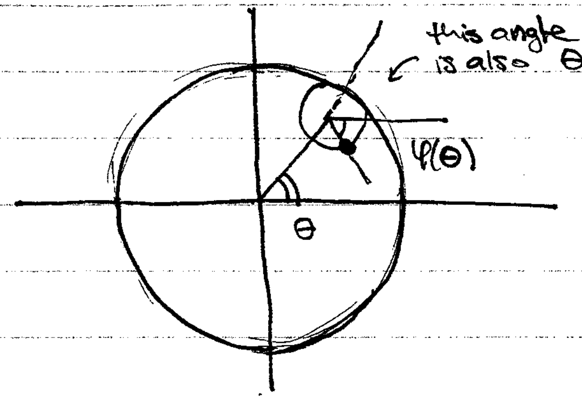
c) $\underline{k=\cancel{3}-1}$: time scales like $\alpha^{3/2}$

distance scales like α

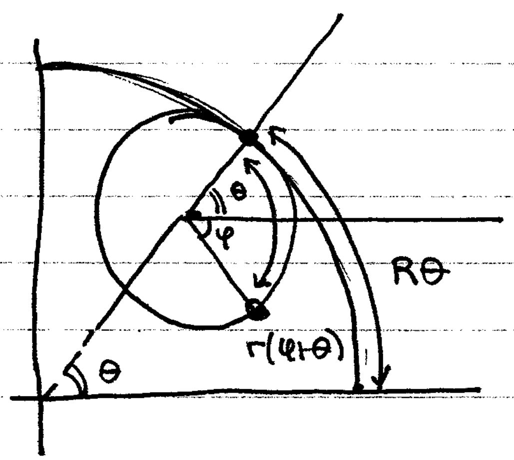
$\Rightarrow t^{\cancel{3/2}\,2/3} \sim d \Rightarrow \boxed{d^3 \sim t^2}$

6. WHEEL IN A HOLE





TO SIMPLIFY THE PROBLEM, DEFINE φ TO BE THE ANGLE OF THE MASS RELATIVE TO THE x-AXIS. THE SYSTEM HAS 1 DOF, SO $\varphi = \varphi(\theta)$



THE DISTANCE ROLLED BY THE WHEEL, $r(\varphi+\theta)$, MUST EQUAL THE DISTANCE TRAVERSED ALONG THE HOLE'S CIRCUMFERENCE $R\theta$. THUS:

$$\varphi = \frac{1}{r}(R-r)\theta$$

THE CARTESIAN COORDINATES OF THE MASS ARE

$$x = \underbrace{(R-r)\cos\theta}_{\text{POSITION OF WHEEL CENTER}} + r\cos\varphi = (R-r)\cos\theta + r\cos\left(\frac{R-r}{r}\theta\right)$$

$$y = (R-r)\sin\theta - \underbrace{r\sin\varphi}_{\text{POS OF MASS REL. TO WHEEL CENTER}} = (R-r)\sin\theta - r\sin\left(\frac{R-r}{r}\theta\right)$$

DISK ROTATES CLOCKWISE

a) CARTESIAN COORDINATES ARE USEFUL SINCE $T = \frac{1}{2}m|\dot{\vec{r}}|^2 = \frac{1}{2}m(\dot{x}^2+\dot{y}^2)$

$$\dot{x} = -(R-r)\sin\theta\,\dot{\theta} - (R-r)\sin\left(\frac{R-r}{r}\theta\right)\dot{\theta}$$
$$\dot{y} = (R-r)\cos\theta\,\dot{\theta} - (R-r)\cos\left(\frac{R-r}{r}\theta\right)\dot{\theta}$$

$$\dot{x}^2 = (R-r)^2\dot{\theta}^2\left(\sin^2\theta + 2\sin\theta\sin\left(\frac{R-r}{r}\theta\right) + \sin^2\left(\frac{R-r}{r}\theta\right)\right)$$
$$\dot{y}^2 = (R-r)^2\dot{\theta}^2\left(\cos^2\theta - 2\cos\theta\cos\left(\frac{R-r}{r}\theta\right) + \cos^2\left(\frac{R-r}{r}\theta\right)\right)$$

$$|\dot{\vec{r}}|^2 = \dot{x}^2+\dot{y}^2 = (R-r)^2\dot{\theta}^2\left(2 - 2\cos\left(\frac{R}{r}\theta\right)\right)$$

$\cos(x+y) = \cos x\cos y - \sin x \sin y$

$$T = \frac{1}{2}m|\dot{\vec{r}}|^2 = \boxed{m(R-r)^2\dot{\theta}^2\left(1-\cos\left(\frac{R}{r}\theta\right)\right)}$$

b) SINCE $E = T$ IS CONSERVED, T IS CONSTANT.

$$\Rightarrow \dot{\theta} = (\text{const})\left[1-\cos\left(\frac{R}{r}\theta\right)\right]^{-1/2}$$