Hand & Finch problem 5.6

This problem shows how the procedure for constructing the Hamiltonian can sometimes break down. The correct answer to the last part is not $H_\theta = 0$, rather, "H_θ is not defined." Even in the most revered text on the subject, by Goldstein, the very same H_θ is pronounced to be "identically zero", though not without cautionary language! Given this state of confusion, here is an attempt to clear things up.

We start with the general one-degree-of-freedom Lagrangian $L(q, \dot{q}, t)$. Making the change of independent variable, from t to θ, we obtain

$$L_\theta(t, q, t', q') = t' L(q, q'/t', t),$$

where the prime indicates a derivative with respect to θ. This formula defines a Lagrangian for a system with two degrees of freedom, $t(\theta)$ and $q(\theta)$, in terms of a Lagrangian with only one degree of freedom, $q(t)$.

The momenta conjugate to t and q, given L_θ, are always defined:

$$
\begin{aligned}
p_t &= \frac{\partial L_\theta}{\partial t'} && (1)\\
&= L + t' \frac{\partial L}{\partial \dot{q}} \left(-\frac{q'}{t'^2} \right) && (2)\\
&= L - \left(\frac{q'}{t'} \right) p_q && (3)\\
&= L - \dot{q}\, p_q && (4)\\
&= -H && (5)\\
& && (6)\\
p_q^\theta &= \frac{\partial L_\theta}{\partial q'} && (7)\\
&= t' \frac{\partial L}{\partial \dot{q}} \left(\frac{1}{t'} \right) && (8)\\
&= p_q && (9)
\end{aligned}
$$

We see that p_q^θ, the momentum conjugate to q for Lagrangian L_θ, is equal to the momentum conjugate to q for the original Lagrangian L. The superscript θ will therefore be dropped in the following equations.

Let's now focus on the content of equations (3) and (8) above:

$$\begin{aligned} p_t + (q'/t')p_q &= L(q, q'/t', t) \\ p_q &= \frac{\partial L}{\partial \dot{q}}(q, q'/t', t). \end{aligned}$$

We have two equations that relate the momenta p_t and p_q to the generalized coordinates t and q and their velocities, but the two velocities always appear in the combination q'/t'. If we were to solve for the momenta we would find

$$p_t = f(t, q, q'/t') \tag{10}$$
$$p_q = g(t, q, q'/t'), \tag{11}$$

where f and g are functions of three variables. Now here is the key observation. In order to define the Hamiltonian H_θ it is necessary to be able to express the velocities in terms of the momenta. The equation

$$H_\theta = p_t t' + p_q q' - L_\theta$$

is not a proper definition unless the t' and q' above (which includes the prefactor t' in L_θ) can be expressed in terms of just the variables t, q, p_t, and p_q! However, if we look at equations (10) and (11) we see that we can only solve for the ratio

$$q'/t' = h(t, q, p_t, p_q)$$

and not for t' and q' separately. In mathematical terms, the transformation from the Lagrangian variables t, q, t', q' to the Hamiltonian variables t, q, p_t, p_q is singular. In physical terms, a single point in phase space would correspond to a continuum of initial conditions for the Lagrangian dynamics (since only the velocity ratio is determined).

The appearance of time t as a *dependent* variable, in addition to the position coordinates x, y, z, is what we expect in a relativistic theory of mechanics. In such a theory we would describe the "world-line" of a particle as four coordinate functions parameterized by an independent variable θ:

$$\begin{aligned} S &= mc^2 \int d\tau \\ &= mc \int \sqrt{c^2 - v^2}\, dt \\ &= mc \int \sqrt{c^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2}\, dt \\ &= mc \int \sqrt{(ct')^2 - x'^2 - y'^2 - z'^2}\, d\theta. \end{aligned}$$

The action S is the integrated proper time along the world-line multiplied by mc^2 in order to have units of action. The primes on the coordinate functions are derivatives with respect to θ, as above. The relativistic Lagrangian for a free particle of mass m is thus

$$L(t,x,y,z,t',x',y',z') = mc\sqrt{(ct')^2 - x'^2 - y'^2 - z'^2}.$$

The conjugate momenta for our relativistic Lagrangian are well defined and because all four coordinates are absent from L we have four conserved momenta:

$$\begin{aligned}
p_t &= \frac{\partial L}{\partial t'} = \frac{mc^2\, t'}{\sqrt{(ct')^2 - x'^2 - y'^2 - z'^2}} = \frac{mc}{\sqrt{1-(v/c)^2}} \\
p_x &= \frac{\partial L}{\partial x'} = \frac{-mc\, x'}{\sqrt{(ct')^2 - x'^2 - y'^2 - z'^2}} = -\frac{m\dot{x}}{\sqrt{1-(v/c)^2}} \\
p_y &= -\frac{m\dot{y}}{\sqrt{1-(v/c)^2}} \\
p_z &= -\frac{m\dot{z}}{\sqrt{1-(v/c)^2}}.
\end{aligned}$$

We recognize these (up to signs) as the four components of the conserved relativistic 4-momentum. This example shows that the conjugate momenta have useful physical interpretations even in cases where the construction of the Hamiltonian breaks down.