Homework 9: **The Last Homework**

Course: Physics 231, *Methods of Theoretical Physics* (2016)
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Due by: Friday, December 2

Important: All homework for grading must be submitted by December 2 in class. No homework will be accepted after this date.

1 Clebsch-Gordon Coefficients

Here is the table of Clebsch-Gordan coefficients for the tensor product of the $\mathbf{4} \otimes \mathbf{3}$ representation of [the Lie algebra of] SU(2). Recall that in this notation, we're labeling the representation by the dimension of the vector space upon which it acts. That is: the spin-3/2 representation is four-dimensional because the in that space can four different J_z eigenvalues, $m = 3/2, 1/2, -1/2, -3/2$.

3/2×1

	5/2 +5/2
+3/2 +1	1

	5/2 +3/2	3/2 +3/2
+3/2 0	2/5	3/5
+1/2 +1	3/5	−2/5

	5/2 +1/2	3/2 +1/2	1/2 +1/2
+3/2 −1	1/10	2/5	1/2
+1/2 0	3/5	1/15	−1/3
−1/2 +1	3/10	−8/15	1/6

	5/2 −1/2	3/2 −1/2	1/2 −1/2
+1/2 −1	3/10	8/15	1/6
−1/2 0	3/5	−1/15	−1/3
−3/2 +1	1/10	−2/5	1/2

	5/2 −3/2	3/2 −3/2
−1/2 −1	3/5	2/5
−3/2 0	2/5	−3/5

	5/2 −5/2
−3/2 −1	1

(a) The direct *product* $\mathbf{4} \otimes \mathbf{3}$ of SU(2) decomposes into a direct *sum* of ordinary (non-tensor) representations[1]. To what direct sum of ordinary representations does this decompose into?

(b) What is the J_3 eigenvalue of the highest weight state in this tensor product?

(c) One basis to describe the states is $|m_1, m_2\rangle$ where m_1 and m_2 are the J_3 quantum numbers of the spin-3/2 and spin-1 representations respectively. The more useful basis is $|j, m\rangle$ which states the irreducible representation j and the J_3 eigenstate within that representation. Write the decomposition of the $|j = 3/2\,,\, m = -1/2\rangle$ state in terms of $|m_1, m_2\rangle$ states.

2 Evidence for SU(3) color

Long before we had our current understanding of elementary particle physics, nuclear physicists were puzzled by the jumble of particles that they kept discovering. Many of these looked like cousins of the proton and neutron. One in particular is the Δ^{++} baryon. The ++ means that it has twice the charge of a proton.

[1] These are usually called 'irreducible representations.'

The Δ^{++} is spin-3/2 with respect to angular momentum. It also is "spin-3/2" with respect to a separate SU(2) called **isospin**[2].

The fact that Δ^{++} is spin-3/2 means that there are four states with different angular momenta in the z-direction $s_3 = 3/2, 1/2, -1/2, -3/2$. Let's focus on the highest angular momentum state.

Even for the highest angular momentum state, there are a bunch of other states related by isospin. These have silly names, like the Δ^-, Δ^0, and Δ^+. Roughly speaking, $\Delta^+ = I_-\Delta^{++}$, $\Delta^0 = I_-\Delta^+$ and so forth, where I_- is the lowering operator of isospin[3]. Just like we focus on only the $s_3 = 3/2$ angular momentum state, let's focus on only the state with highest I_3 which is the Δ^{++} with $I_3 = 3/2$.

At around the time of the discovery of the Δ baryons, people were already thinking about treating these objects as if they were made up of more elementary objects—though few people actually thought that these 'quarks' were more than just mathematical tricks—and so they thought that the Δ^{++} should be thought of as a bound state of three *up quarks*[4]. These up quarks are spin-1/2 and isospin-1/2. The $s_3 = 3/2$ piece of the Δ^{++} is the highest state coming from a tensor product of three up quarks. That is, it is the highest angular momentum state and the highest isospin state:

$$\left| s = \frac{3}{2}, s_3 = \frac{3}{2}; I = \frac{3}{2}, I_3 = \frac{3}{2} \right\rangle = |s_3 =\uparrow, I_3 =\uparrow\rangle\, |s_3 =\uparrow, I_3 =\uparrow\rangle\, |s_3 =\uparrow, I_3 =\uparrow\rangle \ . \tag{2.1}$$

Here's the puzzle. The right-hand side of (2.1) is *symmetric* with respect to the exchange of any two of the up quark states. *However*, we also know that the up quarks have half-integer spin so that the product wavefunction must be *antisymmetric* with respect to the exchange of its constituents. How can this be possible?

Argue that one solution to this problem is the existence of an additional quantum number that distinguishes the up quark states. In particular, the up quarks may be in the fundamental representation ($\mathbf{3}$) of an additional symmetry, $\mathrm{SU}(3)_{\text{color}}$.

The decomposition of the $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}$ tensor product of SU(3) is

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1} \ . \tag{2.2}$$

Which irreducible color representation (on the right-hand side) should correspond to the Δ^{++}? (Hint: we don't observe a multiplicity of Δ^{++} states.) Should this representation be symmetric or antisymmetric with respect to the exchange of elements?

Note that the $\mathbf{10}$ is symmetric, since it came from applying lowering operators which are all symmetric. The $\mathbf{1}$ turns out to be antisymmetric.

3 The Adjoint Representation of SU(3)

The $\mathbf{8}$ is the adjoint representation of SU(3). Show that the eight generators of SU(3) equipped with their commutation relations are, themselves, a manifestation of the adjoint representation. Recall

[2]Treat this as an additional quantum number that commutes (i.e. has nothing to do with) ordinary spin.

[3]Notice that apparently isospin is related to electric charge.

[4]If you're not familiar with the Standard Model of particle physics, you might ask, "what's up quark?" To which I respond, "not much, what's up with you?"

that this means that the generators are both the vectors in the vector space and the operators that act on the vector space. If the T_i are the generators of SU(3), then the operators $d(T_i)$ act on vectors $|T_j\rangle$ via

$$d(T_i)|T_j\rangle = |[T_i, T_j]\rangle = c_{ij}{}^{k}|T_k\rangle \ . \tag{3.1}$$

Use the commutation relations of the SU(3) algebra to show that this furnishes an eight-dimensional representation of SU(3). It's useful to work in the basis where there are two diagonal generators H_1 and H_2 and three pairs of raising and lowering operators. Label the states by their H_1 and H_2 eigenvalues. Plot the states on the (H_1, H_2) plane.

Hint: write out all the commutation relations. (We jotted them down in class, the ones not written are zero.) Then start with the highest weight state, E^3_+, and use the lowering operators to generate the rest of the states. Recall that there should be two states with vanishing H_1 and H_2 eigenvalues.