Homework 8: **Flatland, Lie Algebras**

Course: Physics 231, *Methods of Theoretical Physics* (2016)
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1 Green's Functions on Flatland

There's a fantastic novella written in 1884 called *Flatland: A Romance of Many Dimensions*[1]. It is a story about a 2+1 dimensional universe that doubled as a social commentary. In this problem, we weill explore electrodynamics in such a universe.

1.1 Warm Up: Potentials in Flatland

Suppose Flatland is a sufficiently 'nice' manifold: let it be $\mathbb{R}^{2,1}$, Minkowski space with two spatial dimensions.

Flatland has electrons and electric charge. Unlike our (3+1)-dimensional universe, the electric field only has two components: E_x and E_y. The electromagnetic fields live in a 2-form—as they do in (3+1)-dimensions—that we call F. How many components does the magnetic field have?

Half of Maxwell's equations tell us that $dF = 0$ so that we may write $F = dA$. How many components are in the potential, A? Write this in terms of an electric potential and a vector potential.

1.2 Green's Function: set up

Now we're going to re-do Problem 1 of Homework 6. We would like to understand the propagation of an electromagnetic wave in Flatland. I'll do the first step for you:

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial \mathbf{r}^2}\right]\varphi(\mathbf{r},t) = \rho(\mathbf{r},t) \qquad\qquad \left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial \mathbf{r}^2}\right]\mathbf{A}(\mathbf{r},t) = \mathbf{j}(\mathbf{r},t)\ . \tag{1.1}$$

Convince yourself that the Green's function for each component is (compare to Homework 6):

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial \mathbf{r}^2}\right]G(x,x') = \delta^{(3)}(x-x') \equiv \delta(x-x')\delta(y-y')\delta(t-t')\ . \tag{1.2}$$

Solve for the Green's function. Use the same Fourier transform convention as Homework 6:

$$\widetilde{G}(k,x') = \int d^3x\, e^{ik\cdot x}G(x,x') \qquad\qquad G(x,x') = \int đ^3k\, e^{-ik\cdot x}\widetilde{G}(k,x')\ , \tag{1.3}$$

where we've used the notation $đ = d/2\pi$. Further, $k = (E/c, \mathbf{k})$ and we recall that $k\cdot x \equiv k_\mu x^\mu = Et - k_x x - k_y y$. Also note the δ function in 3-space: $\delta^{(3)}(x-x') = \int đ^3k e^{-ik\cdot(x-x')}$.

[1] https://en.wikipedia.org/wiki/Flatland; see also http://www.geom.uiuc.edu/~banchoff/Flatland/

Answer: It should be no surprise to you at all that you end up with the following:

$$G(x, x') = \int đ^3k \, \frac{c^2}{c^2\mathbf{k}^2 - E^2} e^{-ik\cdot(x-x')} \ . \tag{1.4}$$

Compare this to Homework 6.

1.3 Something familiar

For convenience, write $y \equiv x - x' = (cu, \mathbf{s})$. Check that it is still true that

$$-ik \cdot y = iks\cos\theta - iEu \ . \tag{1.5}$$

Unlike Homework 6, however, our integration measure is different:

$$d^3k = dE\, d^2\mathbf{k} = |\mathbf{k}|\, dE\, d|\mathbf{k}|\, d\theta \ . \tag{1.6}$$

Warning: At this stage, you have plenty of options for which integral to do first. Mathematically, it doesn't matter which order you do them—but some paths have more tedious integrals than others. I'm going to give hints for doing the integrals in the following order: dE, $d|\mathbf{k}|$, $d\theta$.

Go ahead and perform the dE integral for the case of the retarded Green's function. Hint: hey, haven't you done this *exact* integral before? (If you need practice with contour integrals, feel free to do this step from scratch.)

Answer:

$$\int dE \frac{e^{-iEt}}{c|\mathbf{k}|^2 - E^2} = \frac{-i\pi}{c|\mathbf{k}|}\left(e^{ic|\mathbf{k}|t} - e^{-ic|\mathbf{k}|t}\right) \ . \tag{1.7}$$

Comment: At this point, you might want to make a mental note of where $i\varepsilon$'s would be if we were keeping track of them.

1.4 The $d|\mathbf{k}|$ integral

Now perform the $d\theta$ integral. I suggest doing this the following way: combine the result of (1.7) into a sine function. Then use the relation

$$\int_0^\infty e^{iAk}\sin(ckt)\, dk = \frac{ct}{-A^2 + c^2t^2} \ , \tag{1.8}$$

this is a slightly mathematically dubious step which is okay if we judiciously kept track of the $i\varepsilon$. Specifically, this only holds when $t > r|\cos\theta|$. We'll be sloppy about this for now, we'll see the actual constraints come out more cleanly below.

Answer: You should get

$$G(y) = \frac{1}{(2\pi)^2}\int d\theta \frac{ct}{c^2t^2 - r^2\cos^2\theta} \ . \tag{1.9}$$

1.5 The angular integral

A useful trick here is to use

$$\int_0^{2\pi} \frac{d\theta}{1 - B\cos^2\theta} = 2\pi\sqrt{\frac{1}{1-B}} \ . \tag{1.10}$$

This only holds when $B < 1$.

Answer: You should get

$$G(y) = \frac{1}{2\pi}\frac{c}{\sqrt{c^2t^2 - s^2}} \ , \tag{1.11}$$

with the caveat that $c^2t^2 > s^2$.

1.6 Dimensional Reduction

There's actually a much nicer way to get the lower-dimensional Green's function from a higher-dimensional Green's function. Prove that

$$G(x, y, t) = \int_{-\infty}^{\infty} G_{(3)}(x, y, z, t)\, dz \ , \tag{1.12}$$

where $G_{(3)}$ is the (3+1)-dimensional Green's function we derived in Problem 1 of Homework 6. In other words: one can derive the (2+1)-dimensional Green's function by integrating over ('integrating out') the third dimension.

First do this without using the explicit form of $G_{(3)}(x, y, z, t)$. Just use the defining relation that

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial \mathbf{r}^2}\right] G_{(3)}(x - x') = \delta^{(3)}(x - x') \equiv \delta(x - x')\delta(y - y')\delta(z - z')\delta(t - t') \ . \tag{1.13}$$

Hint: This is easiest to do in momentum space. First show that in Fourier space, the left-hand side of the defining relation (1.2) is

$$\left(E^2 - k_x^2 - k_y^2\right) \widetilde{G}(k_x, k_y, E). \tag{1.14}$$

Plug in the ansatz (1.12) and show that this becomes:

$$\left(E^2 - k_x^2 - k_y^2\right) \widetilde{G}_{(3)}(k_x, k_y, 0, E) \ . \tag{1.15}$$

Compare this to the defining relation of the Fourier transform $\widetilde{G}_{(3)}$ (with $k_z = 0$), and show that this directly implies that:

$$\left(E^2 - k_x^2 - k_y^2\right) \widetilde{G}(k_x, k_y, E) = 1 \ , \tag{1.16}$$

thus proving the ansatz (1.2).

1.7 Dimensional Reduction: explicit result

For the retarded Green's function in (3+1)-dimensions, we found

$$G_{(3)}(\mathbf{r}, t) = \frac{c}{4\pi r}\delta(r - ct) \qquad \text{for } t > 0 \;, \tag{1.17}$$

and zero for $t < 0$. Integrate this expression over dz to obtain the (2+1)-dimensional Green's function, $G(x, y, t)$,

$$G(x, y, t) = \frac{c}{4\pi}\int_{-\infty}^{\infty} \frac{1}{\sqrt{x^2 + y^2 + z^2}}\delta\left(\sqrt{x^2 + y^2 + z^2} - ct\right) dz \;. \tag{1.18}$$

Hint: you have one integral to do, but fortunately you have a δ-function to do it. The only problem is that the δ function is of the form $\delta\left(f(z)\right)$. One way to solve this is to do a change of variables to $w = \sqrt{x^2 + y^2 + z^2}$ so that the δ-function is of the form $\delta(w - ct)$. Doing this more generally gives a convenient rule,

$$\delta\left(f(z)\right) = \sum_{z_i} \frac{1}{|f'(z_i)|}\delta(z - z_i) \qquad z_i \text{ such that } f(z_i) = 0 \;. \tag{1.19}$$

Answer: The retarded Green's function is:

$$G(x, y, t) = \begin{cases} \frac{c}{2\pi}\frac{1}{\sqrt{c^2t^2 - x^2 - y^2}} & \text{if } c^2t^2 > x^2 + y^2 \\ 0 & \text{otherwise} \end{cases} \;. \tag{1.20}$$

1.8 The Odd Electrodynamics of Flatland

Plot $G(r, t)$ as a function of $r = \sqrt{x^2 + y^2}$ for two values of t. Compare this to the corresponding plot for $G_{(3)}(r, t)$.

Compare this to Problem 1.7 of Homework 6. Draw the support of $G(r, t)$ on a spacetime diagram. In other words, given a δ-function signal at $r = 0$, $t = 0$, what does the propagation of the signal look like in spacetime? Compare this to the (3+1)-dimensional case.

Comment: Think about how strange this lower-dimensional universe is! In our universe, if you used a flash camera in the dark, the each part of the room would light up for a brief instant corresponding to when the photons from the flash reaches each part of the room. In this lower-dimensional universe, $G(r, t)$ is telling us that the entire room remains illuminated for an *infinite* amount of time with a rapidly decreasing (but never strictly zero) intensity.

2 Left-Invariance of the Lie Bracket

Show that if X and Y are left-invariant vector fields on a Lie group G, then their Lie bracket $[X, Y]$ is also left-invariant.

HINT: you should start by writing out what it is you want to show. In this case, it is

$$L_{a*}[X,Y]_g = [X,Y]_{ag} \; . \tag{2.1}$$

In words: if we take the commutator (Lie bracket) of X and Y at some point $g \in G$ and push it forward to a point $ag \in G$, then does it remain the same?

HINT 2: I suggest taking the left-side of (2.1) and applying it to a test function, $f : G \to \mathbb{R}$. Then with some manipulations, you'll want to show that you end up with the right-side of (2.1) acting on a test function.

HINT 3: an intermediate step is to show

$$(L_{a*}[X,Y]_g)\, f = X_g\,(Y(f \circ L_a)) - Y_g\,(X(f \circ L_a)) \; . \tag{2.2}$$

HINT 4: Here's how you should use the left-invariance of X and Y. You should note that left-invariance means $L_{a*}X_g = X_{ag}$ so that the action on a function is

$$X_{ag} f = (L_{a*}X_g) f = X_g(f \circ L_a) \; . \tag{2.3}$$

HINT 5: Another way to use the left-invariance of, say, Y is

$$(Y(f \circ L_a))\,(g) = Y_g(f \circ L_a) = (L_{a*}Y_g)f = Y_{ag}(f) = (Yf)(ag) = ((Yf) \circ L_a)\, g \; . \tag{2.4}$$

This shows that $Y(f \circ L) = (Yf) \circ L_a$. Make sure you can 'read' what these expressions say. The left-hand side of (2.4) is the directional derivative of a function $(f \circ L_a) : G \to \mathbb{R}$ acting on a point $g \in G$. The function $(f \circ L_a)(g)$ is defined to be a composition, $f\,(L_a(g))$. The expression to the right of the first equality is saying the same thing: $Y_g(f \circ L_a)$ means 'take the directional derivative along Y at the point $g \in G$ of the function $(f \circ L_a)$. Then we're just using definitions of the push-forward and left-invariance.

3 Structure constants

Recall that the structure constants of a Lie group G with generators[2] T_i are given by

$$[T_i, T_j] = c_{ij}{}^{k} T_k \; . \tag{3.1}$$

Using the properties of the Lie bracket (commutator), show that these satisfy:

$$c_{ij}{}^{k} = -c_{ji}{}^{k} \; , \tag{3.2}$$

and further

$$c_{ij}{}^{\ell} c_{\ell k}{}^{m} + c_{jk}{}^{\ell} c_{\ell i}{}^{m} + c_{ki}{}^{\ell} c_{\ell j}{}^{m} = 0 \; . \tag{3.3}$$

[2]Recall that the generators of a Lie group are the basis elements of the Lie algebra, i.e. the basis of the tangent space at the identity, T_eG. We use the common notation that $e = \not{\Bbbk}$.

4 Matrix Lie Algebras

Recall from lecture that matrix Lie groups G are simply described in terms of conditions on their elements, M. One may write $M(t)$ to be a path through G such that $M(0) = \mathbb{1}$. By taking the time derivative on the defining condition of the group G, one finds an expression for the **Lie algebra**, the set of tangent vectors at the origin.

4.1 Generators of the Special Orthogonal group

We showed in class that the group $O(N)$, composed of orthogonal matrices[3] is generated by antisymmetric matrices. There are thus $\frac{1}{2}n(n-1)$ generators. The group $SO(N)$ is composted of orthogonal matrices which, in addition, have unit determinant. What are the generators of $SO(N)$, how many are there? Comment on whether or not this is surprising.

4.2 Generators of the Special Unitary Group

The group $U(N)$ is composed of $N \times N$ unitary matrices, that is complex matrices M such that $M^\dagger M = \mathbb{1}_N$. What types of matrices are the tangent vectors at $\mathbb{1} \in G$ (Lie algebra) of $U(N)$?

$SU(N)$ is composed of unitary matrices which, in addition, have unit determinant. What types of matrices are the tangent vectors at $\mathbb{1} \in G$ (Lie algebra) of $SU(N)$?

Extra Credit

5 Falling Cats

One problem that we didn't get to talk about in this class is the question of how it is that falling cats always land on their feet. I was reminded of this when the *Washington Post* recently had a piece on the falling cat problem[4]. This problem has many connections to geometrical mechanics, group theory, and gauge theory. I encourage you to pursue, in particular:

- "The Square Cat," E. Putterman, O. Raz; Am. J. Phys. 76 1040 (2008), `arXiv:0801.0926`
- "Gauge Kinematics of Deformable Bodies," A. Shapere & F. Wilczek (Wilczek went on to win the Nobel prize in physics for his work in gauge theory)
- "Gauge theory of the Falling Cat," R. Montgomery, Fields Institute Communications, 1993
- "Falling cats, parallel parking, and polarized light" R. Batterman, http://dx.doi.org/10.1016/S1355-2198(03)00062-5

[3] Real $N \times N$ matrices, M, such that $M^T M = \mathbb{1}_N$.

[4] https://www.washingtonpost.com/news/animalia/wp/2016/11/04/scientists-just-cant-stop-studying-falling-