Homework 7: The Cake is a Lie

Course: Physics 231, *Methods of Theoretical Physics* (2016)
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Today's pop culture reference title comes to you by way of *Portal*[1] and Sophius Lie[2], who was a group theorist before group theory was a thing—so Wikipedia lists him as someone who worked on applications of symmetries to geometry and differential equations.

1 Selected Recycled Problems

These are selected problems that were made 'extra credit' in Homework 6.

1.1 Electromagnetic Field of a Moving Charge

Consider the electric field of a point charge at rest. Write the electromagnetic field strength tensor $F_{\mu\nu}(x)$ for this field. Perform a Lorentz transformation to a frame where the point charge has velocity β in the positive x-direction. Determine $F_{\mu\nu}(x)$ in this reference frame. What are the electric and magnetic fields in this boosted frame?

1.2 Hodge duality is electromagnetic duality

Recall the Hodge star operator acting on a k-form living on an n-dimensional space:

$$\star dx^{i_1} \wedge \cdots \wedge dx^{i_k} = \frac{1}{(n-k)!}\sqrt{\det g}\, g^{i_1 j_1} \cdots g^{i_k j_k} \varepsilon_{j_1 \cdots j_n} dx^{j_{k+1}} \wedge \cdots dx^{j_n}. \tag{1.1}$$

Defining $\star F = \frac{1}{2}\widetilde{F}_{\mu\nu} dx^\mu \wedge dx^\nu$, show that $\widetilde{F}^{\alpha\beta} = \frac{1}{2}\varepsilon^{\alpha\beta\mu\nu} F_{\mu\nu}$. (There may be a factor of 2 difference.) Show, further, that the components of $\widetilde{F}^{\alpha\beta}$ are related to those of $F_{\mu\nu}$ by a simple transformation on the electric on the magnetic fields; what is this transformation?

2 Frobenius and the Parallel Parking Problem

Southern California was the birthplace of car culture in the United States[3]. One of the consequences is that Southern Californians are notoriously bad at parallel parking—*fit my car in there? That's impossible!* The point of this problem is to mathematically prove that parallel parking into a space that is strictly greater than the length of your car is possible.

This problem is essentially exercise 11.1 in Stone and Goldbart, the discussion in the book may be useful. Recall from lecture that parking a car is an *anholonomic* system. The configuration space

[1] http://knowyourmeme.com/memes/the-cake-is-a-lie
[2] https://en.wikipedia.org/wiki/Sophus_Lie
[3] Uh... citation needed. What do I look like, a historian?

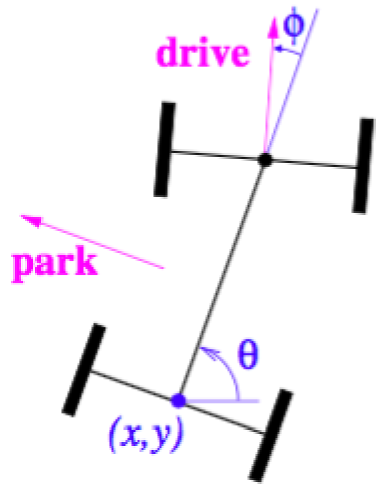

Figure 11.5: Co-ordinates for car parking

Figure 1: Fig. 11.5 from Stone & Goldbart. Coordinates for car parking.

of a car is four-dimensional, given by the position of the car (say the center of mass, or the center of the rear axle), the angle with respect to some reference axis (say, the sidewalk), and the angle of the front wheels used for steering. Call these (x, y, θ, ϕ), as shown in the figure.

A convenient basis of vector fields are:

$$\mathbf{drive} = \cos\phi\left(\cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}\right) + \sin\phi\frac{\partial}{\partial\theta} \tag{2.1}$$

$$\mathbf{steer} = \frac{\partial}{\partial\phi} \tag{2.2}$$

$$\mathbf{skid} = -\sin\phi\left(\cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}\right) + \cos\phi\frac{\partial}{\partial_\theta} \tag{2.3}$$

$$\mathbf{park} = -\sin\theta\frac{\partial}{\partial x} + \cos\theta\frac{\partial}{\partial y}\ . \tag{2.4}$$

Calculate the Lie brackets of all combinations of these vector fields. There are six to calculate.

A driver can only use $\pm$**drive** and $\pm$**steer** to maneuver the car. Use the geometric interpretation ofthe Lie bracket to explain how a suitable sequence of motions (forward, reverse, and turning the steering wheel) can be used to move the car *sideways* into a parking space.

Advice from Burke, *Applied Differential Geometry* (1985, p. 133):

> Just remember to steer, drive, steer back, drive some more, steer, drive back, steer back, drive back: $SDS^{-1}DSD^{-1}S^{-1}D^{-1}$. You have to reverse steer in the middle of the parking place. This is not intuitive, and no doubt is part of the problem with parallel parking. Thus from only two controls you can form the vector fields **steer**, **drive**, **rotate**, and **slide**[4], which span the full tangent space at every point. You can move anywhere in the four-dimensional configuration space.

[4]Burke uses a slightly different set of vector fields than Stone & Goldbart.

If you need more help, you can try looking for discussions on the web. Here are a couple:

- https://www.math.wisc.edu/~robbin/parking_a_car.pdf
- https://rigtriv.wordpress.com/2007/10/01/parallel-parking/ .

3 Group Theory

3.1 Some Tunes!

Okay, first things first. If you're going to do this problem, you're going to need the right music. Head over to YouTube and look up[5] the love song "Finite Simple Group (of Order Two)" by the Klein Four Group (words and music by M. Salomone, Dec. 2004), an a capella group from Northwestern. (Sorry Kamran, it's not prog-rock.)

Try to pick up as many of their corny math puns, especially those related to geometry and group theory. For all the extra credit in the world, come up with a physics love song of comparable cleverness.

If you're looking for something more mellow (and very applicable for the current state of this class) is the "Who Will Grade Your Work" cover of Jewel from Ph.D. comics[6].

3.2 Push-Forwards of Vector fields

Review

Let $\varphi : M \to N$ be a map between two smooth manifolds. Recall that we defined the **push forward** map φ_* that mapped the tangent bundle of M to that of N:

$$\varphi_* : \ TM \to TN \ , \tag{3.1}$$

where the *tangent bundle* is the set of all tangent spaces, e.g. $\{T_pM\}$ for all $p \in M$.

Let $\gamma : \mathbb{R} \to M$ be a curve on M such that $\gamma(0) = p \in M$ and its velocity at p is $\dot{\gamma}(0) = V_p$. Given a function $f : M \to \mathbb{R}$, we can define the action of the vector $V_p \in T_pM$ on f as

$$V_p f = \frac{d}{dt}\left(f \circ \gamma\right)\Big|_{t=0} \ . \tag{3.2}$$

Here $\circ$ means composition, so $f \circ g(x) = f\left[g(x)\right]$. You can convince yourself that this is the same as $d_pf(V_p)$, we're just using an integral curve $\gamma(t)$ of as useful tool.

The push-forward of V_p onto N is φ_*V_p) and has an action on functions on N, $g : N \to \mathbb{R}$, given by

$$(\varphi_* V_p)g = \frac{d}{dt}\left[g \circ (\varphi \circ \gamma)\right]\Big|_{t=0} \tag{3.3}$$

$$= V_p(g \circ \varphi) \ , \tag{3.4}$$

[5] https://youtu.be/BipvGD-LCjU

[6] https://youtu.be/I_YUR6fgWrQ

where in the last line we used the definition (3.2). Make sure you understand all the maps here. (Refer to the diagram we drew in class if necessary.)

The Actual Problem

Suppose M, N, and Q are manifolds with smooth maps $\varphi : M \to N$, $\psi : N \to Q$. Show that

$$(\psi \circ \varphi)_* = \psi_* \circ \varphi_* \ . \tag{3.5}$$

HINT: You should define h to be a test function on Q, that is $h : Q \to \mathbb{R}$. Then consider some point $p \in M$ with some vector V_p attached to its tangent space, T_pM. To prove the above result, you want to show what happens when you push V_p to $T_{\varphi(p)}N$ and then to $T_{\psi\circ\varphi(p)}Q$. Start with

$$\left[(\psi \circ \varphi)_* V_p\right](h) = V_p \left[h \circ (\psi \circ \varphi)\right] \ . \tag{3.6}$$

What do you need to massage the right-hand side into in order to prove the result?

3.3 Left-Invariant Vector Fields on a Lie Group

Now let's return to the case of a Lie Group, G. Recall that this is a group that is also a manifold; that is, it's a group that is continuous.

Review

For $a, g \in G$, we can define a mapping from $G \to G$ called left-translation by a, L_a:

$$L_a g = ag \ . \tag{3.7}$$

This is a nice, differentiable, invertible map. Because G is also a manifold, each point $g \in G$ has a tangent space with tangent vectors. A vector field[7] $X \in TG$ is *left-invariant* if

$$L_{a*}\left(X\big|_g \right) = X\big|_{ag} \ . \tag{3.8}$$

Given a vector $v \in T_{\mathbb{1}}G$ in the tangent space of the origin (identity element), one can construct a unique left-invariant vector field $X(v) \in TG$ with the property that $X(v)_{\mathbb{1}} = v$ using L_{g*}:

$$X(v)\big|_g = L_{g*} v \ . \tag{3.9}$$

The Actual Problem

Show that $X(v)$ is left-invariant. For any point $g \in G$, show that $X(v)\big|_g$ satisfies the condition for being left-invariant.

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[7]By the way, by now it should make sense that vectors live in tangent spaces, $v \in T_pM$ while vector *fields* live in tangent *bundles* TM. These are nice fancy words with a relatively simple idea behind them; their primary use is to impress your friends.