Homework 4: $\mathbb{C}$ Integrals, Sums, and Extra Dimensions

Course: Physics 231, *Methods of Theoretical Physics* (2016)
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Due by: Friday, October 21

Update: 10/17, added a crucial hint for problem 5. Update 10/18: corrected (2.2).

1 Contour Integrals & Infinite Sums

(a) We can analytically continue the ordinary trigonometric functions by letting their arguments take complex values. $\cot \pi z$ is meromorphic on the complex plane with an infinite number of evenly-spaced poles where z is a real integer. Show that these poles are all simple. What are their residues?

(b) Now suppose $f(z)$ is a meromorphic function whose poles do not coincide with the poles of $\cot \pi z$. Consider a contour C which encloses all of the real integers[1] and avoids any of the poles of $f(z)$. Show that

$$\sum_{n\in\mathbb{Z}} f(n) = \frac{1}{2\pi i}\oint_C \pi \cot \pi z\; f(z) dz - \sum_{z_i} \text{Res}(\pi \cot \pi z\; f(z), z_i) \ , \tag{1.1}$$

where z_i runs over the poles of $f(z)$ that are enclosed in C (if any). This is a trick for evaluating a sum over discrete values.

2 Langevin's function as a sum

As an application of (1.1), we now prove the following relation:

$$\coth x - \frac{1}{x} = \sum_{n=1}^{\infty} \frac{2x}{x^2 + n^2\pi^2} \ , \tag{2.1}$$

the expression on the left-hand side is the **Langevin function** and is a limiting case of the **Brillouin function**, which shows up in the calculation of the magnetization of an ideal paramagnet. Define

$$f(z) = \frac{2x}{x^2 + z^2\pi^2} \ , \tag{2.2}$$

and note that we are holding x constant—we have promoted $n \to z$ in the summand above.

(a) Argue that the $\oint_C$ term in (1.1) vanishes when C is taken to be a 'large' contour enclosing the integers $-N, \cdots N$ for large N. Specifically[2],

$$\left| \frac{1}{2\pi i}\oint_C \pi \cot \pi z\; f(z) dz \right| \leq \frac{1}{2}\oint_C |\cot \pi z| \left| \frac{2x}{x^2+z^2\pi^2}\right| |dz| \to 0 \ . \tag{2.3}$$

[1] This can be one big connected contour, or a bunch of little circles around each circle that are connected by 'bridges' which cancel out in a contour integral.

[2] There are fancy ways to do this, but I'm happy arguing that the cotangent is well behaved for $z = Re^{i\theta}$ with $R \gg 1$ and then saying the rest of the integrant goes like dz/z^2 so that it must vanish in the $R \to \infty$ limit.

(b) Now the right-hand side of (1.1) only has the residue term. Show that this evaluates to

$$-\sum_{z_i} \mathrm{Res}(\pi \cot \pi z\ f(z), z_i) = 2i \cot ix = 2 \coth x\ , \tag{2.4}$$

where we recall that the sum is over the residues of $f(z)$.

(c) Finally, rearrange the sum on the left-hand side of 1.1) so that you have a sum from $n = 1$ to $n = \infty$ and hence prove (2.1).

3 The Euler Γ Function & Spheres in d-Dimensions

In class we introduced the Euler Γ function, which can be analytically continued into a meromorphic function

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx\ . \tag{3.1}$$

Following our discussion in class, show that the area of the unit sphere in d dimensions is

$$S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}\ . \tag{3.2}$$

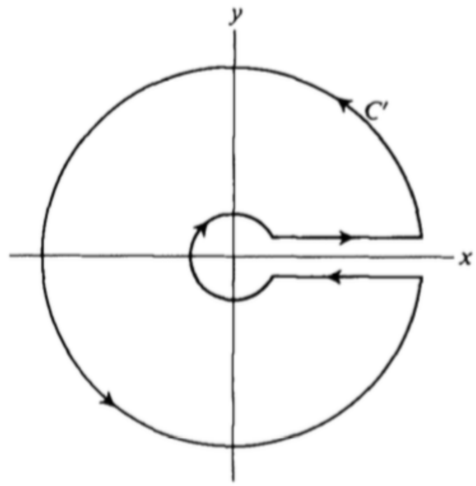
There's always ambiguity (to me at least) when people say things like "d-sphere" ... is this a sphere embedded in d dimensions whose area has units L^{d-1}, or is this the sphere embedded in $(d+1)$ dimension whose area has units L^d? What is the correct interpretation of (3.2)? What is the area of the d-sphere of radius r? What is the [hyper]-volume enclosed by this sphere?

4 Complex Integrals with Branch Cuts

Consider the integral

$$I_\lambda = \int_0^\infty \frac{x^{\lambda-1}}{1+x} dx\ , \qquad 0 < \lambda < 1\ . \tag{4.1}$$

This integrand has a branch cut, which we can take to go along the positive real axis. Evaluate this integral using complex contour integration techniques. You should find $I_\lambda = \pi / \sin \pi\lambda$. HINT: Use the following contour:



As an intermediate step, you should have

$$\left(1 - e^{2\pi i(\lambda-1)}\right) I_\lambda = \frac{-2i \sin \pi\lambda}{e^{-i\pi\lambda}} I_\lambda = 2\pi i \sum_{z_i} \mathrm{Res}\left(\frac{z^{\lambda-1}}{1+z}, z_i\right) , \tag{4.2}$$

where the sum is over poles z_i enclosed in C.

Extra Credit: The beta function is defined as

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt , \tag{4.3}$$

for $x, y > 0$. It turns out that $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$. Consider $B(x, 1-x)$. In the integral definition of this function, make the change of integration variables $t = u/(1+u)$ and show that one ends up with an integral of the form I_λ with $\lambda = x$. Hence show that $\Gamma(1/2) = \sqrt{\pi}$. Using the definition (3.1) and the change of integration variables $x = u^2$, use this result to prove the Gaussian integral (which shows up all over the place in physics),

$$\int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\pi} . \tag{4.4}$$

5 Dimensional Regularization

In physics, one often has to deal with quantifying and understanding apparent divergences in our calculations[3]. A systematic treatment of how to do this is usually buried deep in a course of quantum field theory, where there is a danger of confusing the physics from the mathematical techniques. In this problem we explore one such technique called **dimensional regularization** in a very simple electrostatic system.

Consider the case of an infinite line with constant charge density λ. Pick coordinates such that the charged line is along the y-axis, and the observer is on the x-axis[4]. Observe that the potential (an

[3]These non-analytic results are the sign of our theory trying to tell us something, see problem 9 of homework 3.
[4]By rotational symmetry x may as well be the radial cylindrical coordinate, ρ.

integral over the Green's function!) at x is

$$V(x) = \frac{\lambda}{4\pi} \int_{-\infty}^{\infty} \frac{dy}{\sqrt{x^2+y^2}} \ . \tag{5.1}$$

Note that this integral is divergent[5]. What does this mean and how do we make sense of quantities like $\mathbf{E} = -\nabla V$ in this case? As physicists, we know better than to deal with infinite quantities—instead, we **regulate** them: we *quantify* the infinity by some parameter which allows us to compute physical quantities like $\mathbf{E}$ and check that those physical quantities are independent of the regulator.

Regulate the dy integral (5.1) by converting the one-dimensional integral into a $(1-2\epsilon)$-dimensional integral, where ϵ is a small number. Don't stop to ask what this means[6]. This means we write:

$$V(x) = \frac{\lambda}{4\pi} \int_{0}^{\infty} dy \, d\Omega_{(1-2\epsilon)} \frac{y^{-2\epsilon}}{\mu^{-2\epsilon}} \frac{1}{\sqrt{x^2+y^2}} \ , \tag{5.2}$$

where $d\Omega_{(1-2\epsilon)}$ is the differential area of the unit sphere in $(1-2\epsilon)$ dimensions, the factor of $y^{-2\epsilon}$ is inserted the give the right measure for a $(1-2\epsilon)$-dimensional volume, and some *arbitrary* dimensionful parameter μ has been introduced to make units work out[7].

(a) Show that this integral evaluates to

$$V(x) = \frac{\lambda}{4\pi} \frac{\mu^{2\epsilon}}{x^{2\epsilon}} \frac{\Gamma(\epsilon)}{\pi^{\epsilon}} \ . \tag{5.3}$$

(b) Confirm that as $\epsilon \to 0$, $V(x)$ becomes ill defined. Hint: Recall from $\Gamma(z+1) = z\Gamma(z)$ that $\Gamma(z)$ has a simple pole at the origin.

(c) Calculate the electric field in the x direction; $E_x = -\partial_x V(x)$ and show that

$$E_x = \frac{\lambda}{4\pi} \left[\frac{2\epsilon \mu^{2\epsilon} \Gamma(\epsilon)}{\pi^{\epsilon} x^{1+2\epsilon}} \right] \ . \tag{5.4}$$

(d) Take the $\epsilon \to 0$ limit and observe that $E_x = \lambda/2\pi x$, which is independent of the arbitrary scale μ and behaves the way that one expects.

Hints and extra credit: This problem is worked out in http://arxiv.org/abs/0812.3578; you are strongly encouraged to read and think carefully about the entire article.

Added 10/17: very helpful hint: Use the relation

$$\int_0^{\infty} \frac{y^{n-1} dy}{\sqrt{x^2+y^2}} = \frac{1}{2} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{1-n}{2})}{\sqrt{\pi}} x^{n-1} \ . \tag{5.5}$$

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[5]The expression is also scale invariant: $V(x) = V(kx)$. Think about what this means physically: an infinite line charge looks the same when you're close to it versus when you are far from it. This makes sense: you have no other length scale to compare to—thus by dimensional analysis, you have no basis to even use the words 'close' or 'far.'

[6]https://www.youtube.com/watch?v=ItV8utelYlc. You can (and should) think about what this 'slightly less than one dimension' means after you see the end result.

[7]I am using units such that $[V] = [\lambda]$. You can put in ϵ_0 as appropriate, if you really wish.

Extra Credit

These problems are not graded and are for your edification. You are strongly encouraged to explore and discuss these topics, especially if they are in a field of interest to you.

6 'Hyper-Complex' Numbers

Quaternions are an extension of complex numbers. Introduce two additional imaginary numbers, j and k such that

$$i^2 = j^2 = k^2 = -1 \ . \tag{6.1}$$

Unlike ordinary complex numbers, these additional imaginary numbers do not commute with each other:

$$ij = -ji = k \tag{6.2}$$
$$jk = -kj = i \tag{6.3}$$
$$ki = -ik = j \ . \tag{6.4}$$

Quaternions, just like complex numbers, are a vector space. Show that a basis for this vector space is given by the 2×2 identity matrix and the three **Pauli matrices**,

$$\mathbb{1}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \tag{6.5}$$

Specifically: show that one may write a quaternion $z = a + bi + cj + dk$, with $a, b, c, d \in \mathbb{R}$, as a matrix

$$z \to a\mathbb{1}_2 + b\sigma^1 + c\sigma^2 + d\sigma^3 \ . \tag{6.6}$$

Show that the relations (6.2–6.4) are satisfied. Show further that

$$\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk} e_k \ , \tag{6.7}$$

where ϵ_{ijk} is the totally antisymmetric tensor defined by $\epsilon_{123} = 1$ and the statement that $\epsilon_{ijk} = -\epsilon_{ikj} = \epsilon_{jik}$. Quaternions happen to show up all the time in physics. They also happen to show up often in 3D computer graphics. For those of you who have seen the Pauli matrices in quantum mechanics, this should not be so surprising. Read "The quaternion group and modern physics" by P.R. Girard to get a preview or how these curious objects will appear in the near future of this course[8].

[8] *Eur. J. Phys.* 5 (1984) 25-32. Part of the goal of these problems is for you to get used to using the VPN to access journal articles off-campus.