Homework 3: Complex Analysis

Course: Physics 231, *Methods of Theoretical Physics* (2016)
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Due by: Friday, October 14

Corrected: 10/11, problem 6 $f(z) \to f(1/z)$ on right-hand side. Thanks Cliff.

1 Cauchy–Riemann in silly coordinates

(a) Write the Cauchy–Riemann equations in polar coordinates.

$$f(z) = f(r,\theta) = u(r,\theta) + iv(r,\theta)\,. \tag{1.1}$$

Hint: you want $\Delta f/\Delta z$ as $\Delta z \to 0$. Unlike the Cartesian case, you cannot just take δz along the $\Delta\theta$ direction because that doesn't correspond to $\Delta z \to 0$ by itself.

(b) Show that $f(z) = \ln z$ is analytic for all finite (nonzero) z.

2 How to find Residues

We saw that a **meromorphic** (analytic up to poles) function has a Laurent series expansion about a point z_0,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \,. \tag{2.1}$$

The a_{-1} term has special significance and is known as the **residue** of f at z_0, $\text{Res}(f, z_0)$. When we have an explicit Laurent expansion about z_0, identifying the residue is a matter of reading off the $(z-z_0)^{-1}$ coefficient. Alternately, when z_0 is a simple pole the function can be written as

$$f(z) = \frac{F(z)}{(z-z_0)}\,, \tag{2.2}$$

where $F(z)$ is analytic at z_0. In this case, the residue is $\text{Res}(f, z_0) = F(z_0)$. This leads to the sometimes-useful guide: If $f(z_0)$ is not finite but $(z-z_0)\,f(z)$ is finite, then

$$\text{Res}(f, z_0) = \lim_{z\to z_0} (z-z_0)\,f(z)\,. \tag{2.3}$$

What do we do if $(z-z_0)\,f(z)$ is *not* finite? For example, what if both a_{-1} and a_{-2} were non-zero? How does determine the residue (a_1) in such a case? Explain why the following algorithm works: Find the positive integer m such that $F_m(z) = (z-z_0)^m f(z)$ is finite at $z = z_0$, then the residue is

$$\text{Res}(f, z_0) = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}} F_m(z)\bigg|_{z=z_0}\,. \tag{2.4}$$

3 Harmony of Analytic Functions

If $f(z) = u(x, y) + iv(x, y)$ is analytic in a region, then u and v are both **Harmonic** functions on the plane:

$$(\partial_x^2 + \partial_y^2)u = (\partial_x^2 + \partial_y^2)v = 0 \ .$$

Show that the contour $u(x, y) = c_1$ are perpendicular to the contours $v(x, y) = c_2$ for constants $c_{1,2}$.

4 Contour Integrals

Calculate the following integrals:

(a) $\int_0^\infty \frac{x^2}{x^4 + 16} dx$

(b) $\int_{-\infty}^\infty \frac{\sin x}{x^2 + 4x + 5} dx$

(c) $\int_0^\infty \frac{1}{x^6 + 1} dx$ using a contour including the real line and a large semicircle in the complex plane

(d) $\int_0^\infty \frac{1}{x^6 + 1} dx$ using a that encloses the 'pizza slice' wedge between $\theta = 0$ and $\theta = \pi/3$. Hint: The line integral along the diagonal is proportional to the integral you're trying to calculate.

(e) $P \int_0^5 \frac{1}{x - 3} dx$, where P denotes the **principal value**: avoid the pole at $x = 3$ by integrating from $0 \leq x < 3 - \epsilon$ and $3 + \epsilon < x \leq 5$ for small ϵ. Show that this is ϵ-independent.

(f) $\int_{-\infty}^\infty \frac{1}{ax^2 + bx + c} dx$ for $a > 0$ and $b^2 - 4ac < 0$. Why are these conditions necessary?

5 Integral representation of the step function

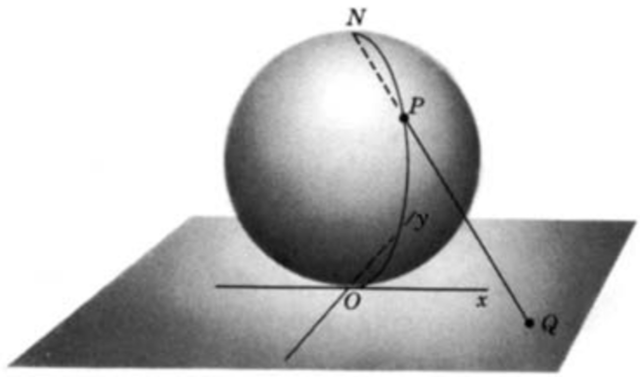
The step function is defined by $\Theta(x) = 0$ for $x < 0$ and $\Theta(x) = 1$ for $x > 0$. Show that this is equivalent to

$$\Theta(x) = \frac{1}{2\pi i} \int_{-\infty}^\infty \frac{e^{ixz}}{z - i\varepsilon} dz \ . \tag{5.1}$$

6 The Riemann Sphere

In class we introduced the idea of the **Riemann sphere**. The Riemann sphere is a completion of the complex plane that include the point at infinity. That is to say, $\infty = Re^{i\theta}$ as $R \to$ 'very big' and for any θ. Note that this means $i\infty \equiv \infty$.

The mapping is demonstrated below (image from Boas, *Mathematical Methods in the Physical Sciences*):



Imagine a sphere situated at the origin, O of the complex plane. The north pole of the sphere, N, is our reference point. Any point on the complex plane Q is mapped to a point P on the Riemann sphere given by the intersection of the line NQ with the sphere. Infinity is identified with N. Observe that an integral along the real line has an integral around the Riemann sphere. Also note that the notion of "inside" versus "outside" becomes somewhat slippery[1]. A useful convention is that 'inside' is to the left of the direction of the contour[2].

(a) Show that the residue of a function f at ∞ is *Corrected $f(1/z)$ on right-hand side, Oct 11. Thanks Cliff.*

$$\text{Res}(f, \infty) = \text{Res}\left(-\frac{f(1/z)}{z^2}, 0\right), \tag{6.1}$$

HINT: we don't understand how to deal with a pole at ∞, so map that pole to a pole that's somewhere on the complex plane. Be sure to explain the sign.

(b) Imagine a small curve C that loops once around a point P somewhere 'in the middle' in the Riemann sphere, as shown above. We know that $f(z) = 1/z$ is analytic in this region, so $\oint_C f(z)dz = 0$. However, if we reverse the orientation of the curve C and follow the 'inside is to the left' rule, this curve now encloses the pole at $z = 0$ which has $\text{Res}(f, 0) = 1$. If we're just taking $C \to -C$, we expect the integral to flip signs. How does this result make sense? HINT: What is the residue of $f(z) = 1/z$ at infinity?

7 Extra Credit

These problems are not graded and are for your edification. You are strongly encouraged to explore and discuss these topics, especially if they are in a field of interest to you.

[1] A physicist and a mathematician are asked to optimize the amount of land inside a fence of fixed length, L. The physicist deduces that enclosing a circular region with radius $R = L/2\pi$ optimizes the area enclosed. The mathematician sees a flaw in this logic and subsequently throws away most of the fence and encloses a circle of size $r \ll R$. The mathematician steps inside tiny region and says, "I declare myself to be on the outside."

[2] Note how *orientation* avoids the issue raised in the above footnote.

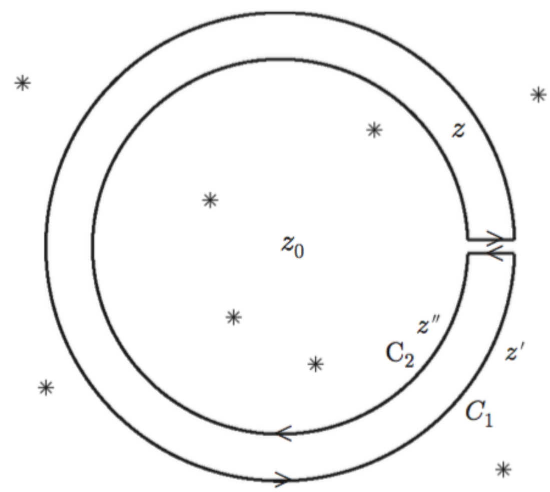
8 Laurent Theorem

Prove the Laurent Theorem for the expansion about z_0 of a function $f(z)$ in a region where it is meromorphic:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \qquad\qquad a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)} dz \ , \tag{8.1}$$

where C is a contour that loops once counter-clockwise around z_0.

Consider the contour below (image from Cahill, *Physical Mathematics*), enclosing an annular region that includes the point z without any poles (asterisks).



Both the outer and inner contours, C_1 and C_2, encircle z_0. Use the **Cauchy theorem**,

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(w)}{w - z_0} dw \ , \tag{8.2}$$

where C encloses a region in which $f(z)$ is *analytic*. Taking $C = C_1 - C_2$, the contour shown above, one has

$$f(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z')}{z' - z} dz' - \frac{1}{2\pi i} \oint_{C_2} \frac{f(z'')}{z'' - z} dz'' \ . \tag{8.3}$$

Consider the following quantities:

$$r(z') = \frac{z - z_0}{z' - z_0} \qquad\qquad R(z'') = \frac{z - z_0}{z'' - z_0} \ . \tag{8.4}$$

Note that $|r(z)| < 1$ and $|1/R(z)| < 1$. Write (8.3) as

$$f(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z')}{(z' - z_0)\,[1 - r(z')]} dz' - \frac{1}{2\pi i} \oint_{C_2} \frac{f(z'')}{(z - z_0)\,[1 - 1/R(z'')]} dz'' \ . \tag{8.5}$$

Use the series expansion

$$\frac{1}{1 - s} = \sum_{n=0}^{\infty} s^n \ , \tag{8.6}$$

for $|s| < 1$. Argue that we may now deform C_1 and C_2 to an intermediate contour C. Show that the result of all this proves (8.2).

9 Hierarchy Problem

Always be aware when your working model of physics becomes non-analytic: often your theory is trying to tell you something important. In this problem we'll think a bit about a singularity that you're all familiar with, then connect it to one of the most glaring problems in particle physics.

The observed mass of the electron is a catastrophe in classical physics. The electron has 'intrinsic' rest energy given by $E_0 = m_0c^2$, where m_0 is some 'intrinsic' mass of the electron which is *not* the observed mass. Instead, the mass includes the energy from the electron's electromagnetic field acting on itself. This Coulomb repulsion gives a contribution to the energy

$$\Delta E_{\text{Coulomb}} = \frac{\alpha}{r_e} \ , \tag{9.1}$$

where r_e is the 'radius' of the electron. Note that this is *not* analytic for a point-like electron! Physics is trying to tell us something. Experiments tell us that $r_e \lesssim 10^{-17}$ cm. This corresponds to $\Delta E \gtrsim 10$ GeV. The observed rest mass (or rest energy) is

$$mc^2 = m_0c^2 + \Delta E_{\text{Coulomb}} = 0.5 \text{ MeV} \ . \tag{9.2}$$

Assuming that r_e is around its experimental limit, this means that

$$m_0c^2 = -9.995 \text{ GeV} \ . \tag{9.3}$$

The negative sign isn't weird—m_0 doesn't have any physical significance, only m has to be positive. However, it is apparently very crucial that m_0 is *tuned* to be within 0.1% of $\Delta E_{\text{Coulomb}}$. Note that $\Delta E_{\text{Coulomb}}$ is *not* arbitrary, it comes from measurements of the properties of the electric field. Its large size and the corresponding tuning in m_0 comes from the singularity as $r_e \to 0$. This tuning seems silly:

$$0.5 \text{ MeV} = (-0.995 + 10.000) \text{ GeV}. \tag{9.4}$$

We can avoid this tuning if the expression (9.1) breaks down for r_e smaller than some cutoff length scale R. What is the order of magnitude of this scale R in centimeters? (For example, for what $r_e = R$ is $\Delta E_{\text{Coulomb}} \approx mc^2$?)

We mentioned in class that (9.1) indeed breaks down due to microphysics: we know that the empty space around the electron is populated by virtual electron and positron pairs which polarize the vacuum. That is, they screen the electron's charge. These virtual particles are predicted by quantum theory. They 'exist' on short time scales related to how much energy they must steal from the vacuum. Recalling that $\Delta t \Delta E \sim \hbar$, what is the characteristic *distance* at which these virtual pairs appear? Recall $d = c\Delta t$, and note that $\Delta E = 2m_ec^2$ since charge conservation requires an e^+e^- pair. (Not that we care about factors of 2 in this order of magnitude estimate). Compare this length scale to the value of R found earlier; observe that quantum fluctuations appear and protect the electron at length scales 100 times larger than needed from the point of view of tuning!

The mass Higgs boson suffers from a similar instability called the Hierarchy Problem. Virtual corrections want to push its mass to a very heavy scale, but somehow it is relatively light at 125 GeV. Like the virtual particles in the above example, one of the favorite candidates for solving this problem is supersymmetry, where the spectrum of particles is doubled and pairs of these particles 'screen' the Higgs mass. The absence of supersymmetry at the Large Hadron Collider requires that any supersymmetric solution to the Hierarchy problem must stomach some degree of tuning. For all the extra credit in the world: find a symmetry principle that extends the known theory of particle physics in such a way that (1) the Higgs mass is shielded and (2) the predictions are not ruled out by current experiments.