Homework 2: **Differential Equations, Green's Functions**

Course: Physics 231, *Methods of Theoretical Physics* (2016)
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Due by: Friday, September 30

Don't worry, the assignment is many pages, but the problems should be tractable. I expect that you'll spend more time reading the problems than actually solving them. *Revised Oct. 4, thanks to Cliff for pointing out typos. Revised Oct. 6, thanks Bryan and Jay for more typo corrections.*

1 Discretized $-d^2/dx^2$ Operator, Part I

[Stone & Goldbart exercise 4.9] Consider the equation

$$\mathcal{O}f(x) = g(x)$$

with $\mathcal{O} = (d/dx)^2$, defined on the interval $0 \leq x \leq 1$ with some given source function $g(x)$. Discretize the differential equation as follows:

- Replace the continuum of x values with a discrete lattice of N points,
$$x_n = \frac{n - 1/2}{N}.$$
- Replace the continuous functions $f(x)$ and $g(x)$ with their values on these lattice points, $f_n = f(x_n)$ and $g_n = g(x_n)$.
- Replace the continuum operator $\mathcal{O}$ with the finite difference operator, A which acts on a discretized function as
$$\sum_n A_{nm} f_m = f_{n+1} - 2f_n + f_{n-1}.$$

(a) Introduce additional points f_0 and f_{N+1} to specify boundary conditions. What do Dirichlet boundary conditions, $f(0) = f(1) = 0$, in the continuum correspond to in the discretization? Write out the matrix A; see the matrices T_1 and T_2 in problem 2 as an example. Note: the vectorized f has components $f_1, \cdots, f_N$. Convince yourself that it is not necessary for A to act on the points f_0 and f_{N+1} that we inserted to define boundary conditions. Is the matrix self-adjoint?

(b) Now impose periodic boundary conditions: $f(0) = f(1)$ and $f'(0) = f'(1)$. Why do we not also require $f''(0) = f''(1)$? Unlike part (a), these boundary conditions require identifying $f_N = f_0$ and $f_1 = f_{N+1}$. Write out the matrix A in this case, point out where it differs from part (a). Is the matrix self-adjoint?

(c) Now consider the matrix

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & -2 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 1 & -2 & 1 \\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The null space of B is the set of vectors f such that $Bf = 0$.

(i) Show that this is given by solutions of $f_{n+1} - 2f_n + f_{n-1} = 0$.

(ii) Note that we don't have to worry about the values of f_0 and f_{N+1}. What boundary condition, if any, is being imposed in the continuum?

(iii) One ansatz to solve for f is $f_n \propto \alpha^n$. Using your answer to part (i), what is the condition on α from $Bf = 0$? This condition is degenerate and only provides one linearly independent solution, even though we expect two for a second order 'differential equation'. Based on what you know about this equation in the continuum, what is the second solution?

2 Discretized $-d^2/dx^2$ Operator, Part II

[Stone & Goldbart exercise 5.6] The $k \times k$ matrices

$$T_1 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & -1 & 2 & -1 & 0 \\ 0 & \cdots & 0 & 0 & -1 & 2 & -1 \\ 0 & \cdots & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$T_2 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & -1 & 2 & -1 & 0 \\ 0 & \cdots & 0 & 0 & -1 & 2 & -1 \\ 0 & \cdots & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

represent two discrete lattice approximations to $(d/dx)^2$ on a finite interval. Note the difference in the kk element.

(a) What are the boundary conditions defining the domains of the corresponding continuum differential operators? They are either Dirichlet ($y = 0$) or Neumann ($y' = 0$).

(b) Verify that

$$\left(T_1^{-1}\right)_{ij} = \min(i,j) - \frac{ij}{k+1} \qquad \text{and} \qquad \left(T_2^{-1}\right)_{ij} = \min(i,j)\,.$$

(c) Find the continuum Green's functions for the boundary value problems approximated by T_1 and T_2. Compare each of the matrix inverses with its corresponding continuum Green's function. Are they similar?

3 Self-Adjointness / Hermiticity

[Stone & Goldbart exercise 4.4] Consider the differential operator $\mathcal{O} = (d/dx)^4$. The **formal adjoint**, $\mathcal{O}^\dagger$, is an operator defined with respect to an inner product $\langle \cdot | \cdot \rangle$ such that

$$\langle f|\mathcal{O}|g\rangle - \langle \mathcal{O}^\dagger f|g\rangle = Q[f,g]\big|_b^a$$

for some functional[1] Q and where a and b are the boundaries of the space.

(a) For the usual L^2 inner product, $\langle f|g\rangle = \int f^*(x)g(x)dx$, what is the formal adjoint $\mathcal{O}^\dagger$? What is the corresponding surface term, $Q[f,g]$? Hint: the name of the game is to integrate by parts. Here's the intermediate step:

$$\frac{d}{dx}\left[f^*g''' - (f')^*g'' + (f'')^*g' - (f''')^*g\right] = f^*g'''' - (f'''')^*g.$$

(b) To determine if an operator is truly adjoint, rather than just formally so, one must specify the domain and the boundary conditions of $\mathcal{O}$ and $\mathcal{O}^\dagger$. Suppose the domain of $\mathcal{O}$ is the set of functions[2] $g(x)$ defined over $x \in [0,1]$ subject to the boundary conditions

$$g(0) = g'''(0) = g(1) = g'''(1) = 0.$$

What conditions must be imposed on the domain of $\mathcal{O}^\dagger$ (the 'ket' functions f in this case) to make the surface term $Q|_0^1$ vanish? You may use the conditions on g imposed from the domain of $\mathcal{O}$. You should find that the boundary conditions on g are not the same as those on f and therefore $\mathcal{O}$ is not truly self adjoint.

4 Wave equation Green's function, Part 1

Consider the wave equation with a source $f(x)$.

$$\frac{d^2}{dx^2}\psi(x) + k^2\psi(x) = f(x)\ .$$

[1] All we're saying here is that the right-hand side is $\int \frac{d}{dx}Q[f(x),g(x)]\,dx = Q[f,g]|_b^a$. See Stone & Goldbart equation (4.17).

[2] Assume square integrability, $g \in L^2[0,1]$ if you're a stickler for this sort of thing.

(a) Solving by completeness: First consider the boundary conditions $\psi(0) = \psi(1) = 0$. Write down the Green's function $G(x, y)$ using the completeness trick with respect to eigenfunctions e_i with eigenfunctions λ_i with respect to the operator in question,

$$G(x, y) = \sum_i \frac{e_i^*(y) e_i(x)}{\lambda_i}.$$

Hint: The eigenfunctions of d^2/dx^2 are $\psi_n(x) = \sqrt{2} \sin(n\pi x)$. What are the eigenfunctions of the wave operator $\mathcal{O} = d^2/dx^2 + \omega^2$?

(b) Dimensional analysis, again: Oops! I just remembered I was doing *physics* and that this is a differential equation for a vibrating string. x should have dimensions of length—how do I rescale the solution to part (a) if the string has length L and the boundary conditions are imposed at $x = 0$ and $x = L$? Write down k in terms of the angular frequency ω and wave speed c. Hint: to help with dimensional analysis, remember that δ functions have units, too.

5 Wave equation Green's function, Part 2

Another way to solve the Green's function equation,

$$\mathcal{O} G(x, y) = \delta(x - y)\,,$$

is to solve the trivial equation $\mathcal{O} G(x, y) = 0$ for $x \neq y$ in the two regions $x < y$ and $x > y$. This gives double the number of undetermined coefficients. Patching together the solutions at $x = y$ gives the additional boundary conditions to solve the complete system.

(a) For the wave equation with a source in problem 4, solve $\mathcal{O} G_<(x, y) = 0$ in the region $0 < x < y \leq L$. Similarly, solve $\mathcal{O} G_>(x, y) = 0$ in the region $0 < y < x \leq L$. Don't forget the boundary conditions $G(0, y) = G(L, y) = 0$. (Understand why $\psi(0) = \psi(L) = 0$ imposes the same condition on the Green's functions.) Your solutions should be defined up to an overall prefactor. The Green's function $G(x, y)$ is now piecewise defined

$$G(x, y) = \begin{cases} G_<(x, y) & \text{if } x < y \\ G_>(x, y) & \text{if } x > y \end{cases}.$$

(b) Integrate the Green's function equation over an infinitesimal sliver, $x \in (y - \varepsilon, y + \varepsilon)$. This gives an equation

$$\frac{dG(y + \varepsilon)}{dx} - \frac{dG(y - \varepsilon)}{dx} = 1\,.$$

Note: For $\varepsilon \to 0$, the k^2 term becomes negligible. Integrate this one more time to obtain

$$G(y + \varepsilon) - G(y - \varepsilon) = 0\,.$$

Use this 'jump' condition on the first derivative of G and the continuity equation on G to fix the coefficients of $G_<$ and $G_>$. Write out the final piece-wise defined Green's function.

6 Wave equation Green's function, Part 3

Now let's play with the same differential equation with a slightly different twist. Let's re-imagine this differential equation as describing a mass on a spring. We'll re-write variables suggestively,

$$\frac{d^2}{dt^2}y(t) + \omega^2 y(t) = f(t) \, .$$

We also impose initial value conditions: $y(0) = y'(0) = 0$. This is different from the boundary condition $\psi(0) = \psi(L) = 0$ in the previous 2 problems. Convince yourself that these boundary conditions impose $G(0, t_0) = dG(0, t_0)/dt = 0$. We'll solve this using yet another technique. As you do each part, think about why the Laplace transform turns out to be a useful tool for this type of problem.

The *Laplace transform* of a function f is

$$F_{[f]}(p) = \int_0^\infty f(t) e^{-pt} dt \, .$$

Convergence requires a lower bound on $\mathrm{Re}(p)$, such as $\mathrm{Re}(p) > 0$, depending on $f(t)$. In this problem we won't have to worry about this.

(a) Calculate the Laplace transform of the Green's function equation,

$$F_{[G'']}(p) + F_{[\omega^2 G]}(p) = F_{[\delta(t-t_0)]} \, .$$

Here G'' means $(d/dt)^2 G(t, t_0)$. You'll have to do some integration by parts. Note why the surface terms vanish: it's a combination of the Laplace transform e^{-pt} factor and the initial value conditions. Show that the result is

$$F_{[G]}(p) = \frac{e^{-pt}}{p^2 + \omega^2} \, .$$

(b) It is not so simple to figure out the inverse Laplace transform; usually one has a table of common transforms in their bag of tricks. Here's one you may have expected to try:

$$F_{\sin(at)}(p) = \frac{a}{p^2 + a^2} \, .$$

Also useful is the rule for functions proportional to a unit step:

$$f(t) = \begin{cases} h(t - t_0) & \text{if } t > t_0 > 0 \\ 0 & \text{if } t < t_0 \end{cases} \qquad \Longrightarrow \qquad F_{[f]}(p) = e^{-pa} F_{[h]}(p) \, .$$

Use these to solve for $G(t, t_0)$ as a piece-wise function.

(c) Comment on the relation of the piece-wise definition to causality.

7 Say hello to Flip

We've decided to cancel most of our discussion sections. If you haven't done so already, that's a good time to swing by Flip's office to say hello and introduce yourself.

Extra Credit

These problems are not graded and are for your edification. You are strongly encouraged to explore and discuss these topics, especially if they are in a field of interest to you.

1 Dimensional Analysis and Why Nature is Second Order

Most differential equations in physics are second order (in derivatives) or less. We observed that this can be understood from the mantra that *physics is local*; higher derivative terms encode non-locality. Here we give a toy example for why nature appears to stop at second order.

Often our favorite differential equations come from a least action principle, $\delta S = 0$. In natural units, the action is dimensionless, $[S] = 0$. We'll take the example of the dynamics of a scalar field, $h(x)$. This is a function defined over all of space and time. Quantum excitations of this field correspond to particles[3].

(a) The action is written as an integral of a Lagrangian density, $\mathcal{L}$ over spacetime: $S = \int d^4x\, \mathcal{L}$. To match onto something familiar, this is just saying that $S = \int dt\, L$ with $L = \int d^3\mathbf{x}\, \mathcal{L}$. In natural units, what is the mass dimension of $\mathcal{L}$?

(b) The Lagrangian density is a function of the field $h(x)$ and derivatives, $\partial_\mu = \partial/\partial x^\mu$. A dynamical particle necessarily has at least the following term in the Lagrangian density,

$$\mathcal{L} \supset \frac{1}{2} h(x) \partial_\mu \partial^\mu h(x) = \frac{1}{2} h(x) \left(\partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2 \right) h(x).$$

This corresponds to the kinetic energy term in the Lagrangian. In natural units, what is the mass dimension of a derivative, ∂_μ? (Does it matter whether it's a time derivative or a space derivative?) What is the mass dimension of the field $h(x)$?

(c) We can ignore terms in the Lagrangian density with fewer than two powers[4] of $h(x)$. Lorentz invariance says that you cannot have a term with an odd number of derivatives, so let's imagine what would happen to a term in $\mathcal{L}$ that had two powers of $h(x)$ and four derivatives, for example:

$$\mathcal{L} \supset \Lambda^n h(x) \partial^4 h(x).$$

We've included an overall prefactor, Λ^n. Assume that Λ has mass dimension of one. Given your answers to the previous question, what is the value of n for which the $\mathcal{O}(\partial^4)$ term has the correct dimension for a Lagrangian density term?

(d) The answer to the above question is $n = -2$. One of the 'deep ideas' in physics is the *renormalization group* (related ideas: effective theories, anomalous scaling.). I hope you have the chance to study this in the many ways it shows up in different types of physics, but one of the punchlines of this idea is that the prefactor Λ^n in operators like the one above should be identified with a *cutoff* for the theory. That is: this is an energy scale (inverse length scale) at which the model you're using to describe the system breaks down because you're not including

[3] Some of you are not familiar with field theory—don't worry, you'll just have to take some of the statements as facts. As 'extra credit,' find colleagues to discuss where some of these facts come from. Bonus points if you can identify where I've oversimplified the story.

[4] Extra credit: think about such terms would mean for the differential equation that comes from $\delta S = 0$.

micro-physics. For example: if you had an action for the behavior of water waves where the water is treated as a continuous medium, then the theory breaks down at the scale where individual water molecules are resolved. Is Λ a 'big' number or a 'small' number? Specify in what sense you mean this—what units are you using, and to what are you comparing? HINT: As you know from Fourier analysis, derivatives bring down powers of momentum. So a more concrete question is: I have a theory of the scalar field $h(x)$ which I know is valid at some energy scale, E—if all my momenta are of this characteristic energy scale, $\partial \to k \approx E$, then is the coefficient of the $\mathcal{O}(\partial^4)$ term in (c) bigger or smaller than the ordinary kinetic term in (b)?

2 Mixed space fermion propagator in a warped extra dimension

The Randall–Sundrum model of a warped extra dimension is a nice playground to explore ideas about compositeness and the holographic principle (AdS/CFT) in particle physics. In such a scenario, one may be interested in the Green's function for a particle to propagate in the extra dimension. For those who are interested, try to follow the manipulations in Appendix F of `https://arxiv.org/abs/1004.2037` starting from the differential equation (F.23). The propagator is a Green's function $\Delta(p, z, z')$ where p is the ordinary 4-momentum and z, z' are the 'to' and 'from' positions in the extra dimension. There are some bells and whistles floating around, but see if you can identify the following:

(a) Which of the methods in problem 4,5, and 6 above did the authors of `1004.2037` use to solve the Green's function?

(b) The differential operator in (F.23) is given by the expressions in (F.22) with $\partial^2 \to -p^2$. This is just a negative number as far as the Green's function equation is concerned. This equation is of Sturm-Liouville form. What is the function $p(x)$?

(c) Appendix F of this paper is really looking for the solution to the matrix differential equation (F.20). Here the γs are matrices in spin space encoding the chiralities of fermions. How did the authors deal with the matrix nature of this differential equation?

(d) From reading the introduction of the paper, try to figure out what problem the authors were trying to solve by figuring out this strange Green's function.

NOTE: This is a difficult problem asking you to dig into a rather technical paper in a field that many of you do not have much background. If you are interested in high energy theory, I encourage you to try figure out qualitative answers to these questions. If you'd like to learn more about some of these ideas, I suggest skimming `https://arxiv.org/abs/1602.04228`.

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