LEC 26: SPACETIME SYM. 28 NOV 2016

TODAY: SPACETIME SYM
WED: GAUGE THEORY & THE ORIGIN OF FORCE
FRI: GEOMETRIC MECH.

I. TRANSLATIONS: $x^\mu \to x^\mu + a^\mu$

one rep:

$$\underset{D(a)}{\begin{pmatrix} 1 & a^\mu \\ & 1 \end{pmatrix}} \underset{|x\rangle}{\begin{pmatrix} 1 & x^\mu \\ & 1 \end{pmatrix}} = \underset{|x+a\rangle}{\begin{pmatrix} 1 & (x+a)^\mu \\ & 1 \end{pmatrix}}$$

II. TRANSLATIONS + LORENTZ

$$\begin{pmatrix} \Lambda & a \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} = \begin{pmatrix} \Lambda & (\Lambda x + a) \\ & 1 \end{pmatrix}$$

MULT. RULE: $D(\Lambda_2, a_2)\, D(\Lambda_1, a_1) = (\Lambda_2 \Lambda_1, \Lambda_2 a_1 + a_2)$

Semidirect product of Lorentz & translations
(they do not commute)

one comp has opposite sign in η!

III **LORENTZ GROUP** : $SO(3,1)$ or $SO(1,3)$

MATRICES s.t. $x^\mu \eta_{\mu\nu} x^\nu$ invariant

η diag$(+,-,-,-)$

$$(\Lambda^\mu{}_\rho \cancel{x^\rho})\eta_{\mu\nu}(\Lambda^\nu{}_\sigma \cancel{x^\sigma}) = \underbrace{\cancel{x^\rho}\eta_{\rho\sigma}\cancel{x^\sigma}}$$

chose indices st we can just peel off

$$\Rightarrow \boxed{\Lambda^T \eta \Lambda = \eta}$$

GENERATORS: traceless hermition

$$\Lambda = e^{itW} \longrightarrow \eta W + W^T \eta = 0$$

$$\eta_{\mu\rho} W^\rho{}_\nu + W^\rho{}_\mu \eta_{\rho\nu} = 0$$

$$\Rightarrow W_{\mu\nu} + W_{\nu\mu} = 0$$

indices of matrix

one rep: $W^{\lambda\sigma} = \underbrace{\omega^{\mu\nu}}_{\text{transf param}} \; (M_{\underline{\mu\nu}})^{\lambda\sigma}$

LABELS WHICH GEN

$$(M^{\mu\nu})_{\rho\sigma} = i\left(\delta^\mu_\rho \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\rho\right)$$

IV GLOBAL PROPERTIES

the lorentz group breaks up into 4 disconnected components.

$$\Lambda^T \eta \Lambda = \eta \xrightarrow{\det} (\det \Lambda)^2 = 1 \Rightarrow \boxed{\det \Lambda = \pm 1}$$

$$\Big\downarrow \; 00 \text{ comp.}$$

$$(\Lambda^0{}_0)^2 - \sum_i (\Lambda^0{}_i)^2 = 1$$

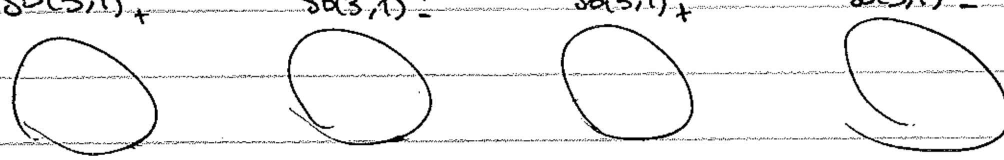
$$\Rightarrow (\Lambda^0{}_0)^2 = 1 + \sum_i (\Lambda^0{}_i)^2 \geq 1$$

$$\Rightarrow \boxed{(\Lambda^0{}_0) \geq 1} \text{ or } \boxed{(\Lambda^0{}_0) \leq 1}$$

ORTHOCHRONOUS

$SO(3,1)^{\uparrow}_{+}$ $\quad SO(3,1)^{\uparrow}_{-}$ $\quad SO(3,1)^{\downarrow}_{+}$ $\quad SO(3,1)^{\downarrow}_{-}$

$\mathbb{1} \in$



det $\Lambda = +$ $\quad$ det $\Lambda = -$ $\quad$ det $\Lambda = -$

$\Lambda^0{}_0 \geq 1$ $\quad \Lambda^0{}_0 \leq 1$ $\quad \Lambda^0{}_0 \leq 1$

$\underbrace{\hspace{10em}}$ <u>not</u> SUBGROUPS.

JUMP BETWEEN COMPONENTS w/

$\Lambda_P = (+,-,-,-)$ PARITY

$\Lambda_T = (-,+,+,+)$ TIME REVERSAL

WE'LL FOCUS ON $SO(3,1)^{\uparrow}_{+}$ SINCE THIS IS THE PIECE YOU CAN GET BY EXPONENTIATING THE ALGEBRA... BUT WE'LL GET TO GLOBAL PROPERTIES...

V. WHY DO SPINORS LOOK LIKE SU(2) VECTORS?

fact: LOCALLY $SO(3,1) \text{ "=" } SU(2) \times SU(2)$
equal w/ caveats

why: LORENTZ ALGEBRA
$$\begin{cases} J_i = \frac{1}{2}\varepsilon_{ijk} M_{jk} & \text{ROTS} \\ K_i = M_{0i} & \text{BOOSTS} \end{cases}$$

$$[J_\cdot, J_\cdot] = i\varepsilon_{\cdot\cdot\cdot} J_\cdot$$
$$[K_\cdot, K_\cdot] = -i\varepsilon_{\cdot\cdot\cdot} J_\cdot$$
$$[J_\cdot, K_\cdot] = i\varepsilon_{\cdot\cdot\cdot} K_\cdot$$

clever trick:
$$A_i = \frac{1}{2}(J_i + iK_i)$$
$$B_i = \frac{1}{2}(J_i - iK_i)$$

in this basis:
$$[A_\cdot, A_\cdot] = i\varepsilon_{\cdot\cdot\cdot} A_\cdot$$
$$[B_\cdot, B_\cdot] = i\varepsilon_{\cdot\cdot\cdot} B_\cdot$$
$$[A_\cdot, B_\cdot] = 0$$

BREAKS UP INTO TWO SEPARATE SU(2)'s!

generaliz of $e^{i\theta}$ vs. e^{δ}

REMARKS

A_i, B_i are not Hermitian ... not compact

whereas: $SU(2) \times SU(2)$ is compact

$\Rightarrow SO(3,1)^{\uparrow}_{+}$ is not "equal" to $SU(2) \times SU(2)$

but: CAN LABEL REPRESENTATIONS OF $SO(3,1)^{\uparrow}_{+}$ by $SU(2) \times SU(2)$ REP.

eg $(\frac{1}{2}, 0)$, $(0, \frac{1}{2})$, $(1, 0)$, ...

SPIN-1/2 wrt A

SPIN-1/2 wrt B

etc.

So electrons are in $(\frac{1}{2},0)$, $(0,\frac{1}{2})$ for example.

antiparticles

what about vector rep? $(\frac{1}{2},\frac{1}{2})$ contains a 4

how to see?

$\sigma^{\mu}{}_{\alpha\dot{\alpha}}$

REMARK : actual relation

$$\mathcal{L}_{\mathbb{C}}[so(3,1)] \cdot \mathcal{L}_{\mathbb{C}}[SU(2)\times SU(2)]$$

complex linear combinations of generators

VI. INTERESTING FACT:

COMPLEXIFICATION OF $SU(2)\times SU(2) = SL(2,\mathbb{C})$

CLAIM : $so(3,1) \cong SL(2,\mathbb{C})/\mathbb{Z}_2$

isomorphic

redundancy by 2

HOW TO SEE: REP OF $so(3,1) \longleftrightarrow SL(2,\mathbb{C})$

$$\sigma^\mu = \left(\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \sigma^i \right) \quad \text{BASIS OF } SL(2,\mathbb{C})$$

$$X \longrightarrow X_\mu \sigma^\mu = \begin{pmatrix} x_0 + x_3 & x_1 - i x_2 \\ x_1 + i x_2 & x_0 - x_3 \end{pmatrix}$$

$so(3,1)$ transf : $X \longrightarrow \Lambda X$ =

$SL(2,\mathbb{C})$ transf : $x_\mu \sigma^\mu \rightarrow N^\dagger (x_\mu \sigma^\mu) N$

OBSERVE : N & (-N) generate the SAME TRANSF.

(N & -N map onto same Λ)

$\hookrightarrow SL(2,\mathbb{C})$ is a double cover of $SO(3,1)$

VII NATURE CARES ABOUT $SL(2,\mathbb{C})$ MORE THAN $SO(3,1)$

$\hookrightarrow$ electrons (& stuff)
are spinors $\leftrightarrow$ reps of $SL(2,\mathbb{C})$

but spacetime appears to have
symmetry $SO(3,1)$

why isn't everything based on vectors
& tensors of vectors?

WHY SHOULD $SL(2,\mathbb{C})$ BE MORE IMPORTANT?

$\rightarrow$ it is simply connected : can reach every element by exp of ALG!
(UNLIKE $SO(3,1)$)

why? fact: POLAR DECOMP:

$$SL(2,\mathbb{C}) \ni g = U e^{h}$$

UNITARY — HERMITIAN (note: no i!)

$$U = \begin{pmatrix} d+ie & f+ig \\ -f+ig & d-ie \end{pmatrix} \quad \text{s.t. } \underbrace{d^2+e^2+f^2+g^2=1}_{S^3}$$

$$h = \begin{pmatrix} c & a-ib \\ a+ib & -c \end{pmatrix} \longrightarrow \mathbb{R}^3$$

so $SL(2,\mathbb{C})$ is topologically $\underbrace{\mathbb{R}^3 \times S^3}_{\text{BOTH SIMPLY CONNECTED}}$

VIII this leads to the idea of a <u>UNIVERSAL COVER</u>

for a given Lie group, $\exists$ unique minimal simply connected group that "covers" it

the Lorentz group is covered by $SL(2, \mathbb{C})$

<u>PROJECTIVE</u> REPRESENTATIONS

rather than $\underbrace{D(g_1)\,D(g_2) = D(g_1 g_2)}_{\text{ORDINARY REP}}$

$$= \underbrace{e^{i\phi(g_1, g_2)}}_{\text{overall phase}} D(g_1 g_2)$$

since $|\psi\rangle \to e^{i\phi}|\psi\rangle$ doesn't change physics.

this is what we see in the SPINOR REP: 720° ROTATION GOES TO SAME STATE.

IX FIELD THEORY

UNITARITY IS SOMETHING WE SHOULD TALK ABOUT. A & B ~~ARE~~ GEN. NONUNIT. TRANS.

b/c K (boosts) WAS ANTIUNITARY
↳ not compact
↑
SO(3,1) vs. SO(4)

REP of noncompact group ⟷ ∞ dimensional (eg translations)

PARTICLES ARE NOT ∞ DIM!
↳ electron: [illegible] SPIN UP, SPIN DOWN

turns out: combine ∞ dim. of Boosts
w/ ∞ dim of translations
(ie REP of full Poincaré group)
gives a way to classify
particles w/ finite dim "spin" reps
[massive & massless cases differ]

∞ identified w/ continuum of p^μ
~~[illegible]~~ that corresp to same particle in different frames -