LEC 25: SHORTCUTS W/ GROUP THY — 24 NOV

↳ game of indices.

NO CLASS WED — happy Thanksgiving

LAST WEEK — IRREDUCIBLE REP (highest wt → lower)

- IR-REP OF SU(2)
- TENSORS OF SU(2) : $\underline{2} \otimes \underline{2} = \underline{3} \oplus \underline{1}$
- SU(3) & TENSORS : $\underline{3} \otimes \underline{\bar{3}} = \underline{8} \oplus \underline{1}$

$\underline{3} \otimes \underline{3} \otimes \underline{3} = 10 + 8 + 8 + 1$

A CHEAP WAY TO DO THIS:

RECALL: SYM & ANTISYM TENSORS TRANSFORM SEPARATELY ← this is exactly what irreducible means!

∀ TENSOR T_{ij}, CONSIDER

$$\frac{1}{\sqrt{2}}\left(T_{ij} \pm T_{ji}\right) \to \frac{1}{\sqrt{2}}\left(R_{ik}R_{jl}T_{kl} \pm R_{jl}R_{ik}T_{lk}\right)$$

$$= R_{ik}R_{jl}\,\frac{1}{\sqrt{2}}\left(T_{ij} \pm T_{ji}\right)$$

$\mathbb{R}$ version of SU(N)

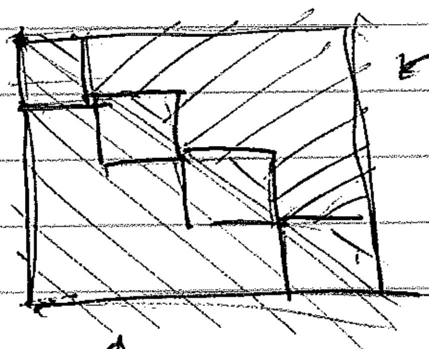
APPLY THIS TO SO(N)

A <u>2-INDEX TENSOR</u> HAS N^2 COMPONENTS.

SOME OBJECT OF PHYSICAL SIGNIFICANCE THAT TRANSFORMS UNDER SO(N)

eg. <u>THEORY</u> IS SO(N) SYMMETRIC, BUT <u>STATE</u> ISN'T.

$$T \to \begin{cases} A_{ij} = \frac{1}{2}(T_{ij} - T_{ji}) & \frac{1}{2}N(N-1) \\ S_{ij} = \frac{1}{2}(T_{ij} + T_{ji}) & \frac{1}{2}N(N+1) \end{cases}$$



$\frac{1}{2}N(N-1)$

$\frac{1}{2}N(N+1)$

IN FACT: S_{ij} IS ITSELF FURTHER REDUCIBLE

$$\sum_i S_{ii} \to \sum_{ikl} \underline{R_{ik}} R_{il} S_{kl} = \delta_{kl} S_{kl} = S_{kk}$$

$(R_{ki})^T R_{il}$

$(R^{-1})_{ki} R_{il}$

① transforms separately

② doesn't transform

singlet ($\underline{1}$)

SO WE CAN DECOMPOSE

$$S_{ij} = \tfrac{1}{2}(T_{ij} + T_{ji}) \to \begin{cases} \underbrace{S_{ij} - \frac{1}{N}\delta_{ij} S_{kk}}_{\text{traceless, symm.} \ \tilde{S}, \ \frac{1}{2}N(N+1)-1} \\ S_{kk} \ \leftarrow \text{trace} \end{cases}$$

SO: FOR $N=3$

A_{ij} has $\frac{1}{2}3\cdot 2 = 3$ components
$\tilde{S}_{ij}$ has $\frac{1}{2}3\cdot 4 - 1 = 5$ components
S_{ii} has 1 component

SO: $\underbrace{3 \otimes 3}_{\text{of } so(3)} = 5 \oplus 3 \oplus 1$

WE CAN UNDERSTAND THIS USING <u>INVARIANT TENSORS</u>

$$S_{ii} = S_{ij}\delta_{ij}$$

$R_{ik}R_{jl}\delta_{kl} = \delta_{ij}$
INVARIANT!
THIS IS B/C $R^T R = \mathbb{1}$
(PART OF OUR DEF. OF GROUP.)

OTHER INVARIANTS?

(S)O(N) ⇒ $\det \underset{R}{\cancel{M}} = 1$

fact: $\varepsilon_{pqr\ldots s} \det \overset{R}{\cancel{M}} = \varepsilon_{ijk\ldots m} \overset{R}{\cancel{M}}_{ip} \overset{R}{\cancel{M}}_{jq} \overset{R}{\cancel{M}}_{rk} \cdots \overset{R}{\cancel{M}}_{sm}$

$\varepsilon_{pqr\ldots s}$ — totally antisymm. tensor.

then: $\varepsilon_{p\ldots s} = \varepsilon_{i\ldots m} R_{ip} \cdots R_{sm}$

so $\underbrace{\varepsilon_{p\ldots s}}_{N}$ is an invariant.

★ THE INVARIANTS δ_{ij} & $\varepsilon_{i_1\ldots i_N}$ ARE SHORTCUTS FOR DECOMPOSING TENSORS

note: YOU ONLY HAVE $\varepsilon_{i_1\ldots i_N}$ ← set by (SO(N))
YOU DO NOT ALSO GET TO
USE ε_{ij}, ε_{ijk}, ...

REMARK: $\varepsilon_{i_1\ldots i_N}$ IS THE TOOL FOR DUALITY.

$F_{\mu\nu} \xrightarrow{\text{dual}} \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} = \tilde{F}^{\mu\nu}$

↑ MAGNETIC FIELD STR.

SO ACTING ON T_{ij} W/ δ_{ij} GAVE TRACE.

$T_{ij}\,\varepsilon_{ij\cdots k}$ GIVES $(N-2)$ COMP. ANTISYMMETRIC TENSOR.

expect this to be the antisymmetric part of T.

eg. for $N=4$:

$T_{ij}\,\varepsilon_{ijkl} = \tilde{T}_{kl}$, totally antisymmetric.

BUT WHAT ABOUT $N=3$?

$T_{ij}\,\varepsilon_{ijk} = \tilde{T}_{k}$ ← one component?

BUT: 3 DOF

cf. 3 DOF OF $(T_{ij}-T_{ji})$

MORE GENERALLY:

$T_{ij}\,\varepsilon_{ij\cdots i_N}$ HAS ${}_N C_{N-2}$ COMPONENTS

$\frac{1}{2!}\frac{N!}{(N-2)!} = \frac{1}{2}(N)(N-1)$ ✓

~~REMARK 1~~:

So: <u>for $N=3$</u> & 2-index tensor

$$T_{ij} \to \varepsilon_{ijk} T_{ij} = \tilde{T}_k \text{, encodes } T_{ij} - T_{ji}$$
$$\delta_{ij} T_{ij} = T_{ii} \text{, trace}$$
$$S_{ij} - \tfrac{1}{N}\delta_{ij} T_{ii} = \tilde{S}_{ij} \text{, sym traceless}$$

& "leftover"

$$3 \times 3 = 5 \oplus 3 \oplus 1$$

this is kind of funny
3D rotations have a rep where
5 objects transform into themselves.

Sound familiar?

$$I_{ij} = \int d^3x\, \rho(x)\, [x^i x^j - \tfrac{1}{3}\delta^{ij} x^2]$$

moment of inertia tensor!

you can use $\varepsilon_{i_1 \cdots i_N}$ & δ_{ij} to simplify larger tensors ... usually we don't have to do anything too crazy.

eg on hw: $3 \times 3 \times 3 = 10 \oplus 8 \oplus 8 \oplus 1$ of <u>SU(3)</u>
the 1 is $\varepsilon_{ijk} q^i q^j q^k$ — <u>totally antisym.</u>

BACK TO AN OLD THEME: FUNCTION SPACES $\leftrightarrow$ VEC SPACES

$SO(3) \leftrightarrow$ ANGULAR MOMENTUM

generators: $L_x = \underbrace{i(z\partial_y - y\partial_z)}$, ...

REP AS DIFFERENTIAL OP.
PREVIOUSLY,

$\frac{1}{2}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$, OR OTHER MATRICES

these satisfy $[L_x, L_y] = iL_z$, etc
in the sense of a Lie Bracket

EIGENFUNCTIONS: $L^2 \; Y_\ell^M(\theta,\varphi) = \ell(\ell+1) \; Y_\ell^M(\theta,\varphi)$
$L_z \; Y_\ell^M(\theta,\varphi) = M \; Y_\ell^M(\theta,\varphi)$

in fact: $Y_\ell^M(\theta,\varphi) = \underbrace{N_\ell^M}_{\text{norm.}} e^{iM\varphi} \underbrace{P_\ell^M(\cos\theta)}_{\text{ASSOCIATED LEG.}}$

SPECIFICALLY, $P_\ell^{M=0}$ IS THE LEGENDRE POLYNOMIAL

shows up when taking moments

$$\frac{1}{|\underline{r}-\underline{r}'|} = \frac{1}{r}\sum_{\ell=0}^{\infty}\left(\frac{r'}{r}\right)^\ell P_\ell(\cos\theta)$$

in the abs. sense: what $SO(3)$ sees.

↓

ILLUMINATING THE VECTOR ⟷ FUNCTION CORRESP.

CONSIDER UNIT VECTOR V

$$V^3 = \cos\theta \longrightarrow P_1(\cos\theta)$$

SYMM. TRACELESS TENSOR

$$T^{ij} = V^i V^j - \tfrac{1}{3}\delta^{ij}|\underline{V}|^2$$

$$T^{33} = \cos^2\theta - \tfrac{1}{3} \longrightarrow P_2(\cos\theta)$$

SYM TRACELESS 3 TENSOR

$$T^{ijk} = V^i V^j V^k - \tfrac{1}{5}(\delta^{ij}V^k + \dots)$$

$$T^{333} = \cos^3\theta - \tfrac{3}{5}\cos\theta \longrightarrow P_3(\cos\theta)$$

↑

RECALL: potential $\Phi = \int d^3s \, \frac{\rho(s)}{|\underline{r} - \underline{s}|}$

$$= \dots + \int d^3s \, \rho(s) \frac{1}{r}\left(\frac{s}{r}\right)^2 P_2(\hat{r}\cdot\hat{s}) + \dots$$

$$\dots + \frac{\hat{r}_i \hat{r}_j}{2r^2} \int d^3s \; \underbrace{s^2 \left(3\hat{s}_i\hat{s}_j - \delta_{ij}\right)}_{\text{QUADRUPOLE TENSOR}}$$

↓

irrep of rotations

HOW THIS IS SHAKING OUT:

↓ gens

<u>SYMMETRY</u> $[H, G] = 0$

REP OF SYM

FIN DIM VEC SP

INFINITE DIM FUNC SP.

tensor rep → irred. rep decomp

EIGENFUNCTIONS

(can expand anything in eigenfn.)

→ well def transf properties

BTW: THIS IS ALL A STRUCTURE ON PHASE SPACE

UNDERLYING MATH IS GEOMETRY

B/C $[H, G] = 0$, these eg func are GOOD WAVEFUNCTIONS

further, DYNAMICAL EQS WILL ONLY ACT W/IN AN <u>IRREP/EIGENF</u>.

you can build Green's functions out of them! :)