LEC 23: TENSOR REPS

11/16

6 MORE LECTURES, NO HW3

② SU(3)
③ SU(3) TENSORS
④ } POINCARE
⑤
⑥ GAUGE THY

we'll do one more HW
ALL HW DUE DEC 2
(last lec.)

TENSOR PRODUCT REPRESENTATIONS

"ADDITION OF ANGULAR MOMENTUM"

MATHEMATICAL QUESTION

GIVEN REP OF $\mathcal{L}(G)$ ON VECTOR SPACES V_1 & V_2, WHAT ~~ARE~~ IS REP OF $\mathcal{L}(G)$ ON $V_1 \otimes V_2$?

PHYSICAL QUESTION (example)

AN ELECTRON ORBITS A PROTON. BOTH ARE SPIN-1/2 PARTICLES. (ASSUME S-WAVE → no ANGULAR ORBITAL MOMENTUM) WHAT IS THE ANGULAR MOMENTUM OF THE ATOM?

what are the states?

we know: $\frac{1}{2} \otimes \frac{1}{2}$ GIVEN BY:

$$\left.\begin{array}{l} |\uparrow\uparrow\rangle \\ |\uparrow\downarrow\rangle \\ |\downarrow\uparrow\rangle \\ |\downarrow\downarrow\rangle \end{array}\right\}$$

is this a rep?

where: $|\uparrow\downarrow\rangle = |\tfrac{1}{2}\rangle \otimes |-\tfrac{1}{2}\rangle$

REP ↓ STATE W/IN REP ↓

SET UP: $V_1 : \quad |j_1, m\rangle$
$V_2 : \quad |j_2, n\rangle$

REPS: $J^{(1)}$ ACTING ON V_1
$J^{(2)}$ —"— V_2

REP ON $V_1 \otimes V_2$ is

$$\boxed{J_i = J_i^{(1)} \otimes \mathbb{1} + \mathbb{1} \otimes J_i^{(2)}}$$

eg. $$J_3 |\uparrow\downarrow\rangle = \left(J_3^{(1)} |\uparrow\rangle\right) \otimes |\downarrow\rangle + |\uparrow\rangle \otimes \left(J_3^{(2)} |\downarrow\rangle\right)$$
$$= \tfrac{1}{2} |\uparrow\downarrow\rangle - \tfrac{1}{2} |\uparrow\downarrow\rangle = 0$$

so $|\uparrow\downarrow\rangle$ has no SPIN angular momentum in the $\hat{z}$ direction.

CHECK: the J_i defined this way satisfy the su(2) commutation relations

eg use $[J_+^{(1)} \otimes \mathbb{1}, \mathbb{1} \otimes J_-^{(2)}] = 0$

(i) OBSERVE: $J_3 |m_1, m_2\rangle = \boxed{(m_1 + m_2)} |m_1, m_2\rangle$

(ii) DEGENERACIES: $|\uparrow\downarrow\rangle$ has same J_3 as $|\downarrow\uparrow\rangle$

DEP ON REP!

RECALL: $J_-|m\rangle = N_m|m-1\rangle$ $\quad \boxed{N_M^2 = \frac{(j+m)(j-m+1)}{2}}$

$J_+|m-1\rangle = N_m|m\rangle$

HIGHEST WEIGHT STATE IS UNIQUE: $|m_1 = j_1, m_2 = j_2\rangle$

eg $|\uparrow\uparrow\rangle$

THEN WE JUST DO WHAT WE DID BEFORE:
APPLY LOWERING OPERATORS

$$\underbrace{J_-|\uparrow\uparrow\rangle}_{m=1} = \left(J_-^{(1)}\otimes 1\!\!1 + 1\!\!1\otimes J_-^{(2)}\right)|\uparrow\uparrow\rangle$$

$$= \underbrace{\left(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle\right)}_{m=0}\mathcal{N}$$

we won't worry about this

$$J_-\left(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle\right) = \left(J_-^{(1)}\otimes 1\!\!1 + 1\!\!1\otimes J_-^{(2)}\right)\left(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle\right)$$

$$= |\downarrow\downarrow\rangle\,\mathcal{N}'$$

$m = -1$

SO WE HAVE A COMPLETE REP!

$|\uparrow\uparrow\rangle$, $|\downarrow\uparrow + \downarrow\uparrow\rangle$, $|\downarrow\downarrow\rangle$ ← SPIN-1 REP

$M=1 \qquad M=0 \qquad M=-1$

WE'RE STILL LEFT w/ $|\downarrow\uparrow - \uparrow\downarrow\rangle$

$$\left.\begin{aligned} J_3|\downarrow\uparrow - \uparrow\downarrow\rangle &= 0 \\ J_\pm|\downarrow\uparrow - \uparrow\downarrow\rangle &= 0 \end{aligned}\right\} \text{SPIN-}\underset{\text{"SCALAR"}}{\underset{\uparrow}{0}}\text{ REP.}$$

WHAT WE'VE DISCOVERED: $\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$

the tensor product of 2 spin-$\frac{1}{2}$ states (eg $e^- p^+$ in s-wave) decomposes into a spin-1 & spin-0 rep.

these reps are <u>totally separate</u>!

SPIN-1 REP HAS <u>TOTAL ANGULAR MOMENTUM</u> = 1 (ie highest weight state)

SPIN-0 REP HAS TOTAL ANG. MOMENTUM = 0. SO THESE STATES DO NOT MIX!

<u>VOCAB</u>: SPIN-1 IS A 3-COMPONENT <u>MULTIPLET</u> in a spin-1 rep, no matter which state you're in, you can do a rotation to go to the $m = \pm 1, 0$ state.

If you're in the spin-0 multiplet, you have $M=0$ & ARE STUCK THERE. NO ROTATION WILL TAKE YOU TO A DIFFERENT ~~HIGHER~~ M. NO ROTATION WILL MIX YOU INTO THE $|j=1, M=0\rangle$ STATE.

OFTEN WE LABEL REPS OF SU(N) BY THEIR DIMENSION, SO SPIN $\frac{1}{2} \leftrightarrow \underline{2}$

SO OUR DECOMPOSITION READS: $\boxed{\underline{2} \otimes \underline{2} = \underline{3} \oplus \underline{1}}$

LET'S DO ANOTHER: SPIN 1 $\otimes$ SPIN $\frac{1}{2}$ — we'll be sloppy w/ norms.

HIGHEST WEIGHT STATE:

$$\left| j = \tfrac{3}{2}, m = \tfrac{3}{2} \right\rangle = \left| \overset{m_1}{1}, \overset{m_2}{\tfrac{1}{2}} \right\rangle$$

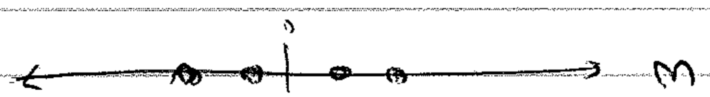
2 BASES: (j, m) (total), (m_1, m_2)

APPLYING LOWERING OPS GIVES:

$$\left| j = \tfrac{3}{2}, m = \tfrac{1}{2} \right\rangle \sim \left| 0, \tfrac{1}{2} \right\rangle \text{``}+\text{''} \left| 1, -\tfrac{1}{2} \right\rangle$$

$$\left| j = \tfrac{3}{2}, m = -\tfrac{1}{2} \right\rangle \sim \left| -1, \tfrac{1}{2} \right\rangle \text{``}+\text{''} \left( \left| 0, -\tfrac{1}{2} \right\rangle \text{``}+\text{''} \left| 0, -\tfrac{1}{2} \right\rangle \right)$$

$$\left| j = \tfrac{3}{2}, m = -\tfrac{3}{2} \right\rangle \sim \left| -1, -\tfrac{1}{2} \right\rangle$$



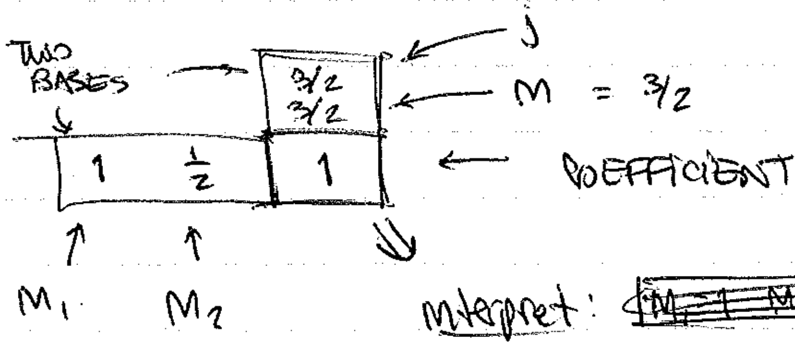
REMAINING STATES

$$\left.\begin{array}{l} \left| 1, -\tfrac{1}{2} \right\rangle - \left| 0, \tfrac{1}{2} \right\rangle \\ \left| -1, +\tfrac{1}{2} \right\rangle - \left| 0, -\tfrac{1}{2} \right\rangle \end{array}\right\} \text{ form } j = \tfrac{1}{2} \text{ REP}$$

AW. BUT THAT WAS REALLY MESSY, ALSO,
I DON'T TRUST YOUR NORMALIZATIONS OF
THE RAISING & LOWERING OPS!
(you shouldn't — $J_-^{(1)}$ & $J_-^{(2)}$ in general
have different normalizations!)

TWO OPTIONS ① DERIVE YOURSELF
② CLEBSCH - GORDON TABLE

eg. $1 \otimes \frac{1}{2}$ (what we just did)



interpret: ~~$|M_1=1\ M_2=\frac{1}{2}\rangle = 1\ |j=\frac{3}{2}$~~

$$|j=\tfrac{3}{2}\ M=\tfrac{3}{2}\rangle = \sqrt{1}\ |M_1=1, M_2=\tfrac{1}{2}\rangle$$

NEXT PART OF TABLE

		3/2	1/2	j
		1/2	1/2	$M = 1/2$
1	−1/2	1/3	2/3	
0	1/2	2/3	−1/3	
M_1	M_2			

$$\left|j=\tfrac{3}{2}\ M=\tfrac{1}{2}\right\rangle = \sqrt{\tfrac{1}{3}}\left|1,-\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{2}{3}}\left|0,\tfrac{1}{2}\right\rangle$$

$$\left|j=\tfrac{1}{2}\ M=\tfrac{1}{2}\right\rangle = \sqrt{\tfrac{2}{3}}\left|1,-\tfrac{1}{2}\right\rangle - \sqrt{\tfrac{1}{3}}\left|0,\tfrac{1}{2}\right\rangle$$

observe: these are normalized
& are orthogonal
(AS THEY MUST BE)

		3/2	1/2	j
		−1/2	−1/2	$m = -1/2$
0	−1/2	2/3	1/3	
−1	+1/2	1/3	−2/3	

$$\left|j=\tfrac{3}{2}\ M=\tfrac{1}{2}\right\rangle = \sqrt{\tfrac{2}{3}}\left|0,-\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{1}{3}}\left|-1,\tfrac{1}{2}\right\rangle$$

$$\left|j=\tfrac{1}{2}\ M=-\tfrac{1}{2}\right\rangle = \sqrt{\tfrac{1}{3}}\left|0,-\tfrac{1}{2}\right\rangle - \sqrt{\tfrac{2}{3}}\left|-1,\tfrac{1}{2}\right\rangle$$

THE LAST STEP IS CLEAR

		$3/2$
		$-3/2$
-1	$-1/2$	1

$$\left| j = \tfrac{3}{2} \quad m = \tfrac{-3}{2} \right\rangle = \left| -1, -\tfrac{1}{2} \right\rangle$$

CHECK : 1×1

		2	1	0	j
		0	0	0	$M = 0$
1	-1	$1/6$	$1/2$	$1/3$	
0	0	$2/3$	0	$-1/3$	
-1	1	$1/6$	$-1/2$	$1/3$	
m_1	m_2				

$|m_1 = 0 \quad m_2 = 0\rangle$ IS A
STATE W/ $M = 0$,
& SO IS $|j = 1, \ M = 0\rangle$,
BUT THEY HAVE NO OVERLAP

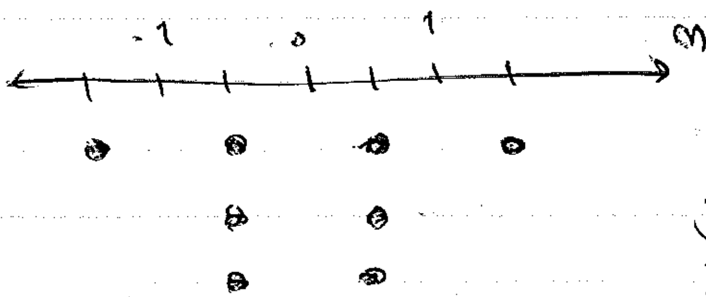
WHAT ABOUT BIGGER TENSOR REPS?

CAN DO IT PAIRWISE

OR THE <u>LONG WAY</u>

↳ qualitative features are easy

eg $\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2}$



$j = 3/2$ "all plus signs"

$j = 1/2$ "1 rel sign"

$j = 1/2$ "2 rel signs"