LECTURE 22: SU(2) 11/14

REVIEW: GROUPS ⟷ SYMMETRIES

ALGEBRA ⟷ TANGENT SPACE OF GROUP

↳ INFINITESIMAL TRANSF.

BASIS: GENERATORS OF TRANSF.

finite transf: $\exp(\theta_a T^a)$

"symmetry of ___"

SU(2): UNITARY 2x2 MATRICES w/ UNIT DETERMINANT

ALGEBRA: ~ PAULI MATRICES

↳ LOOKS LIKE ALGEBRA OF SO(5)

⇝ SUSPECT: DOES SU(2) HAVE TO DO w/ ROTATIONS?

REPRESENTATION OF SU(2) ↙ k-DIMENSIONAL VEC SP.

↳ SPECIFY SIZE OF REP $k \in \mathbb{Z}$

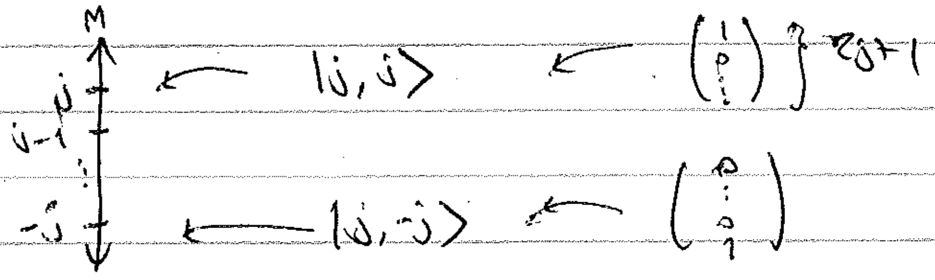
THEN REP IS CALLED (SPIN-j) w/ $j = k/2$

VECTOR SPACE HAS ELEMENTS

$$|j, m\rangle$$

j ↑ $k/2$

m ↑ LABELS WHERE YOU ARE ON A "LADDER" OF STATES



HOW DO <u>INFINITESIMAL TRANSFORMATIONS</u> ACT ON THIS VECTOR SPACE?

$$\text{generators } \overset{(2j+1)\times(2j+1)\text{ MATRIX} \leftarrow \text{UNITARY}}{J_i} \text{ satisfying } \boxed{[J_i, J_j] = i\varepsilon_{ijk} J_k}$$

↳ compare to $[\sigma_a, \sigma_b] = 2i\varepsilon_{abc}\sigma_c$

s.t. $\frac{1}{2}\sigma_i$ IS A REP OF J.

OR IN OTHER WORDS, $J_a = d(\sigma_a)$

REORGANIZE:

$$\boxed{\begin{aligned} &J_3 \\ &J_\pm = \frac{1}{\sqrt{2}}(J_1 \pm i J_2) \\ \\ &[J_3, J_\pm] = \pm J_\pm \\ &[J_+, J_-] = J_3 \end{aligned}}$$

↑

ENCODES EVERYTHING FOR SU(2) (as ALGEBRA, at least)

MORE GENERAL ALGEBRAS:

- DIAGONAL ELEMENTS
- PAIRS OF RAISING/LOWERING

J_3: DIAGONAL. LABEL STATES BY J_3 EIGENVALUE, m.

$$J_3 |\underset{\uparrow \text{ other q'm \#s}}{\cdots}, m\rangle = m|\cdots, m\rangle$$

COMMUTATION RELATIONS:

⇒ $J_\pm$ gives state w/ shifted J_3 eigenval
ie these move you up & down a ladder of states

eg SPIN 1/2 REP:

$$J_3 = \begin{pmatrix} 1/2 & \\ & -1/2 \end{pmatrix}$$

$$J_+ = \begin{pmatrix} & 1 \\ 0 & \end{pmatrix} \qquad J_- = \begin{pmatrix} & 0 \\ 1 & \end{pmatrix}$$

takes $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ takes $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

<u>ANNIHILATES</u> $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

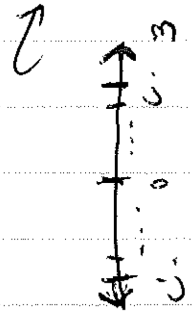
dim vec sp

<u>finite dim rep</u>: ∃ maximum m: $j = \ell/2$

↳ j LABELS <u>REPRESENTATION</u> ⟷ DIMENSIONALITY
m LABELS ONE OF THE $(2j+1)$ STATES IN THAT REP.

THE $J_\pm$ DON'T GENERALLY LOWER YOU INTO A NORMALIZED STATE.

START W/ MAX WEIGHT STATE $|j\rangle$



I will write $|\underset{\substack{\uparrow\\ \text{STATE}}}{m}\rangle$ for $|\overset{\substack{\text{REP}\\ \downarrow}}{j}, \overset{\substack{\text{STATE}\\ \downarrow}}{m}\rangle$

So $|j\rangle$ defined by $J_+|j\rangle = 0$

we know $J_-|j\rangle = N_j\,|j-1\rangle$

↑ for some norm. N_j

HOW TO FIND N_j?

$$\| N_j\,|j-1\rangle \|^2 = |N_j|^2 \langle j-1 | j-1 \rangle$$

$$= (\langle j | J_+)\, J_- |j\rangle$$

↑ using $(J_-)^\dagger = J_+$

why? UNITARY REP

AS ALGEBRA, WE KNOW $T_\pm = (T_\mp)^{-1}$

FOR UNITARY MATRIX $J_\pm = d(T_\pm)$,

$(J_\pm)^{-1} = (J_\pm)^\dagger$

the only trick we have

$$= \langle j | \underbrace{[J_+, J_-]} + \underbrace{J_- J_+}_{0} | j \rangle$$

$$= \langle j | J_3 | j \rangle$$

$$= j \langle j | j \rangle$$

ORTHONORMAL BY DEF

if the bra & ket had other quantum numbers, then these show up as δ_{AB}.

So: $$|N_j|^2 \underbrace{\langle j-1 | j-1 \rangle}_{\text{WANT THESE NORMALIZED BY DEF}} = j \underbrace{\langle j | j \rangle}_{1}$$

$$\Rightarrow \boxed{N_j = \sqrt{j}}$$

Also: $$J_+ |j-1\rangle = J_+ \frac{1}{N_j} J_- |j\rangle$$
$$= \frac{1}{N_j} [J_+, J_-] |j\rangle$$
$$= j/N_j |j\rangle$$
$$= N_j |j\rangle$$

$$J_- |j\rangle = N_j |j-1\rangle$$
$$J_+ |j-1\rangle = N_j |j\rangle$$

& LADDER

WHAT ABOUT NORMALIZATIONS?

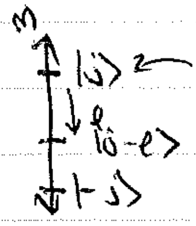
(we know $J_- |j\rangle = \underbrace{\boxed{\sqrt{j}}}_{N_j} |j-1\rangle$

what about $J_- |j-1\rangle$? & so forth.

what is N_{j-1} , N_{j-2} ?

CONSIDER A (not normalized) state

$$(J_-)^\ell |j\rangle \quad \sim (\#) | j-\ell\rangle$$



← highest state: we know it is normalized (by choice)

ALL WE CAN DO TO NORMALIZE $|j-\ell\rangle$ IS TO GO UP THE LADDER TO CONNECT TO $|j\rangle$!

$$J_+ (J_-)^\ell |j\rangle = \Big(\underbrace{[J_+, J_-]}_{J_3} + J_- J_+\Big)\underbrace{(J_-)^{\ell-1} |j\rangle}_{\sim (\#)\, |j-(\ell-1)\rangle}$$

$$= \Big[\overbrace{j-(\ell-1)} + J_- J_+\Big] (J_-)^{\ell-1} |j\rangle$$

RECURSION! GIVES

$$J_-\big[j-(\ell-2) + J_- J_+\big](J_-)^{\ell-2}$$

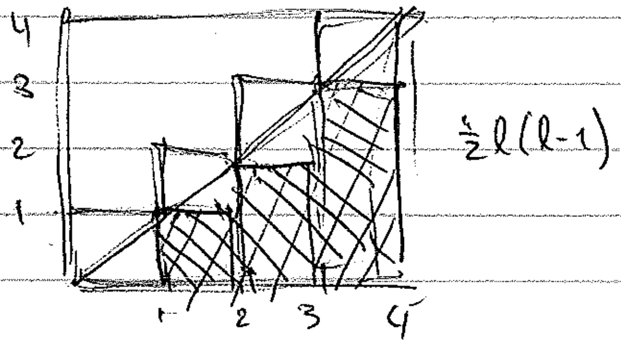
$$= \Big[\big(j-(\ell-1)\big) + \big(j-(\ell-2)\big) + J_-^{\,2} J_+\Big](J_-)^{\ell-3}$$

KEEP GOING UNTIL J_+ HITS $|j\rangle$.

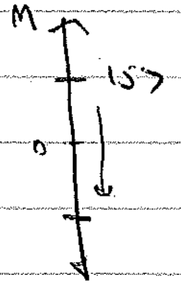
So : $$J_+ (J_-)^l |j\rangle = \Big(j - (l-1) \\ + j - (l-2) \\ + j - (l-3) \dots \\ + j - 1 \\ + j \Big) (J_-)^{l-1} |j\rangle$$

$$= \boxed{\left(lj - \tfrac{1}{2} l(l-1)\right) (J_-)^{l-1} |j\rangle}$$

$$l\left(j - \tfrac{1}{2}(l-1)\right)$$



ok. that's a usefull step.



We also know $$J_- \underbrace{(J_-)^k |j\rangle}_{\cancel{\text{LAST}}\ \text{LOWEST STATE}} = 0$$

$$0 = J_+ \overset{\text{KILLS IT}}{J_-} \overbrace{(J_-)^k |j\rangle}^{\text{LOWEST STATE}}$$

$$= \left(\underbrace{[J_+, J_-]}_{J_3 = (j-k)} + \underline{J_- J_+) \; (J_-)^k |j\rangle} \right.$$

$$\hookrightarrow (J_-) K (j - \tfrac{1}{2}(k-1)) (J_-)^{k-1} |j\rangle$$

SAME TRICK ALWAYS.

$$= \left[(j-k) + k\left(j - \tfrac{1}{2}(k-1)\right) \right] (J_-)^{k-1} |j\rangle$$

$$= \boxed{\tfrac{1}{2}(k+1)(2j-k)} \; (J_-)^{k-1} |j\rangle$$

$$= 0 \quad \Rightarrow \quad \boxed{k = 2j}$$

dim of rep
related to highest
<u>spin</u> of rep.

CONSIDER NORM OF $(J_-)^\ell$

$$\langle j|(J_+)^\ell (J_-)^\ell |j\rangle$$

$$= \langle j|\, J_+^{\ell-1} \left(\underset{\overset{\nearrow}{J_3 = (\ell-1)}}{[J_+, J_-]} + \underbrace{J_-J_+ \big)\, J^{\ell-1} |j\rangle}_{\downarrow \; J_- (\ell-1)(j-\frac{1}{2}(\ell-2))J_-^{\ell-2}|j\rangle} \right.$$

$$= \left(\langle j|\, J_+^{\ell-1}\right) (\ell-1)(j-\tfrac{1}{2}\ell+2) \left(J_-^{\ell-1}|j\rangle\right)$$

$$= \underbrace{\frac{\ell-1}{2}(2j-\ell+1)}\, \| (J_-)^{\ell-1}|j\rangle \|^2$$

$$= \boxed{N_{j-\ell} \cdot N_{j-(\ell-1)} \cdots\cdots N_j}$$

[illegible]

RESULT. $J_-|M\rangle = \frac{1}{\sqrt{2}}\sqrt{(j+M)(j-M+1)}\; |M-1\rangle$

$J_+ |M-1\rangle = \text{—〃—}\; |M\rangle$

CLAIM: GEORGI 3.35

$$N_{j-\ell} \cdots N_j = \frac{(2j)!\,\ell!}{2^\ell (2j-\ell)!}$$

Georgi:

$$J_- |j-\ell\rangle = N_{j-\ell} |j-\ell-1\rangle$$
$$J_+ |j-\ell-1\rangle = N_{j-\ell} |j-\ell\rangle \quad \leftarrow \text{why?}$$

$$N_{j-\ell}^2 = \langle j-\ell | \underset{\underset{\underbrace{[J_+ , J_-]}_{J_3 = (j-\ell)} + J_- J_+}{\uparrow}}{J_+ J_-} |j-\ell\rangle$$
$$= (j-\ell) + \langle j-\ell | J_- J_+ |j-\ell\rangle$$
$$= (j-\ell) + N_{j-\ell+1}^2$$

RECURSION:

$$N_j^2 - 0 = j$$
$$N_{j-1}^2 - N_j^2 = j-1$$
$$\vdots \qquad\qquad \vdots$$
$$N_{j-\ell}^2 - N_{j-\ell+1}^2 = j-\ell$$

$$N_{j-\ell}^2 = (\ell+1)j + \tfrac{1}{2}\ell(\ell+1)$$
$$= \tfrac{1}{2}(\ell+1)(2j-\ell).$$