LEC 21: REPRESENTATIONS

11/8

LAST TIME: GROUPS: ABSTRACT MATHEMATICAL DESCRIPTIONS OF SYMMETRIES.

LIE GROUPS: CONTINUOUS GROUPS (have parameters characterizing the symmetry transform.)

GROUPS THAT ARE MANIFOLDS

LIE ALGEBRA: ELEMENTS OF T_eG w/ A "MULTIPLICATION": COMMUTATOR.

contains all of the local information about the group

can think of this as set of "velocities" @ $e = \mathbb{1}$; give trajectories to reach (almost) any group element.

~~REPRESENTATION~~ ~~SET OF~~ (OR ALGEBRA)
REPRESENTATION : ~~MATRIX~~ ~~MATRICES~~
MAP BETWEEN GROUP ELEMENTS & MATRICES THAT PRESERVES MULTIPLICATION.

eg $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ is the defining (FUNDAMENTAL) rep. of $SO(2)$

$D(g(\theta))$ D is the rep.

CAN ALSO IMAGINE $\begin{pmatrix} \cos\theta & \sin\theta & \\ -\sin\theta & \cos\theta & \\ & & 1 \end{pmatrix}$

for small parameter $\theta = \varepsilon$

$$D[g(\varepsilon)] = \mathbb{1} + i\varepsilon \underbrace{d(T)} + \cdots$$

Taylor exp about $\mathbb{1}$.

REPRESENTATION OF THE ALGEBRA

$$D[g(\theta)] = \lim_{k\to\infty}\left(1 + i\frac{\theta}{k}T\right)^k = e^{i\theta T}$$

$d(T)$

(EXPONENTIAL PARAMETERIZ)

SO OFTEN, WE ONLY NEED TO UNDERSTAND ALGEBRA.

ONE STEP MORE COMPLICATED: ROTATIONS

$SO(3)$: $\mathcal{L}(SO(3))$ is given by

$$T_1 = \begin{pmatrix} 0 & & \\ & 0 & -1 \\ & 1 & 0 \end{pmatrix} \qquad \text{ROT ABOUT } x$$

$$T_2 = \begin{pmatrix} & & 1 \\ & 0 & \\ -1 & & \end{pmatrix} \qquad \text{ROT ABOUT } y$$

$$T_3 = \begin{pmatrix} 0 & -1 & \\ 1 & 0 & \\ & & 0 \end{pmatrix} \qquad \text{ROT ABOUT } z$$

COMPARE TO "DUMB REP" OF $SO(2)$!

$$\text{w/} \quad [T_a, T_b] = \varepsilon_{abc} T_c \qquad \left(\textstyle\sum_c\right)$$

do not commute!

$$\boxed{e^{i\theta^a T_a}\, e^{i\varphi^b T_b} \neq e^{i(\theta^a + \varphi^a) T_a}}$$

INFINITESIMALLY (locally), IT IS TRUE THAT

$$e^{i\theta^a T_a} e^{i\varphi^b T_b} = e^{i\delta^c T_c}$$

FOR SOME δ^c

BCH FORMULA : $e^X e^Y = \exp\left[X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] + \frac{1}{12}[[X,Y],Y] \dots\right]$

you can look it up.

COMMUTATOR!

SO ALGEBRA INDEED ENCODES PROPERTIES OF GROUP MULTIPLICATION.

symmetry in QM acting on 2-state sys? (or more gen?)

EXAMPLE: SU(2) ← special unitary 2×2 matrices

$\det U = 1$ $\quad U^\dagger U = \mathbb{1}$

WHAT IS THE ALGEBRA?

$$\det U = 1 \leftarrow \tfrac{1}{2}\varepsilon_{ab}\varepsilon_{cd} U_{ac} U_{bd}$$

$$\frac{d}{dt}\det U\Big|_{\mathbb{1}} = 0 \leftarrow \tfrac{1}{2}\varepsilon_{ab}\varepsilon_{cd}\left(\dot{U}_{ac}U_{bd} + U_{ac}\dot{U}_{bd}\right)$$

δ

end up w/ $\boxed{\mathrm{tr}(\dot{U}) = 0}$

$$U^\dagger U = \mathbb{1}$$

$$\frac{d}{dt}U^\dagger U\Big|_{\mathbb{1}} = 0 = \dot{U}^\dagger \overset{\mathbb{1}}{U} + U^\dagger \dot{U} = 0$$

$$= \boxed{\dot{U}^\dagger + \dot{U} = 0}$$ ANTI HERMITIAN

cf $so(2)$ ⟶ or $i\dot{U}$ is HERMITIAN

So: ALGEBRA: traceless, (anti) hermitian 2×2

HOW MANY ARE THERE? 8 $\mathbb{R}$ comp.

traceless: -1

(anti) hermitian: -4

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YOU PROBABLY ALREADY KNOW A NICE BASIS OF SUCH MATRICES:

PAULI MATRICES

$$\sigma_1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$$

$$\sigma_2 = \begin{pmatrix} & -i \\ i & \end{pmatrix}$$

$$\sigma_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

$$[\sigma_i, \sigma_j] = 2i\,\varepsilon_{ijk}\,\sigma_k$$

exact same commutation relation as so(3)!

DEF $\boxed{T_i = -\frac{i}{2}\sigma_i}$

$$\mathcal{L}(SU(2)) = \mathcal{L}(SO(3))$$

NB: $\boxed{SU(2) \neq SO(3)}$

BUT ONE THING WE CAN GLEAN FROM THIS IS THAT SU(2) HAS SOMETHING TO DO WITH ROTATIONS.

LOCAL ELEMENTS OF A REP OF SU(2) ARE EXPONENTIATION OF A REP OF $\mathcal{L}(SU(2))$.

IMPORTANT: σ_3 IS DIAGONAL.

SO, e.g. IF WE'RE IN A BASIS WHERE H IS DIAGONAL THEN ACTING W/ $d(\sigma_3)$ or $D(e^{i\sigma_3\Delta\phi})$ WILL NOT CHANGE YOUR EIGENSTATE.

Rep of SU(2)

$J_3 = i d(T_3)$

$J_\pm = \frac{i}{\sqrt{2}}[d(T_1) \pm i\, d(T_2)]$ ← raising/lowering

Georgi, ch 3 summary

$[J_i, J_j] = i\varepsilon_{ijk} J_k$

m ↕

$[J_3, J_\pm] = \pm J_\pm$ ← $J_\pm = \frac{1}{\sqrt{2}}(J_1 \pm i J_2)$

$[J_+, J_-] = J_3$

$$\boxed{\begin{aligned} &J_3 J_\pm |m\rangle - J_\pm J_3 |m\rangle = \pm J_\pm |m\rangle \\ &\Rightarrow J_3 (J_\pm |m\rangle) = (m \pm 1)(J_\pm |m\rangle) \end{aligned}}$$

basis of V: eigenstates

$J_3 |m\rangle = m|m\rangle$

HIGHEST WEIGHT: j : $J_+ |j\rangle = 0$

PROPERLY NORMALIZED STATE

LOWER: $J_- |j\rangle = N_j |j-1\rangle$

normaliz.

$$\underbrace{\langle j | (J_-)^\dagger J_- | j\rangle}_{} = \boxed{N_j^* N_j \langle j-1 | j-1\rangle}$$

$(J_-)^\dagger$: J_+

$\langle j | J_+ J_- | j\rangle = \langle j | J_+ J_- - \underbrace{J_- J_+}_{=0} | j\rangle$

$= \langle j | J_3 | j\rangle$

$= j \underbrace{\langle j | j\rangle}_{\text{normalized}}$

$\Rightarrow N_j = \sqrt{j}$ NORMALIZES $|j-1\rangle$

IF OTHER DIM #'s: $|j\rangle \to |j, \alpha\rangle$, $\langle j | j\rangle \to \delta_{\alpha\beta}$

WHAT ABOUT $J_+ |j-1\rangle$?

$$= \frac{1}{N_j} J^+ J_- |j\rangle$$
$$= \frac{1}{N_j} [J_+, J_-] |j\rangle$$
$$= (j/N_j) |j\rangle$$
$$= N_j |j\rangle$$

CAN CONTINUE THIS PROCESS

$$J_- |j-1\rangle = N_{j-1} |j-2\rangle$$
$$\vdots$$

CHECKING NORMALIZATIONS:

$$N_M = \frac{1}{\sqrt{2}} \sqrt{(j+M)(j-M+1)}$$

BUT: ALSO EXISTS A <u>BOTTOM</u> OF LADDER

$$J^- |j-\ell\rangle = 0$$

$$\Rightarrow J^- |j-\ell\rangle = N_{j-\ell} |j-\ell-1\rangle = 0$$
$$\Rightarrow \uparrow = 0$$

LADDER j IS $2j$ LON

$$N_{j-\ell} = \frac{1}{\sqrt{2}} \sqrt{(2j-\ell)(\ell+1)} = 0 \Rightarrow \boxed{\ell = 2j}$$

$$\Rightarrow \boxed{j = \ell/2}$$

THE STANDARD NOTATION

LABEL STATES BY MAX J_3 (j)
↳ encodes total # of states
(size of representation)

AND BY J_3 (M)

$$\langle j, M' \,|\, J_3 \,|\, j, M \rangle = \delta_{M'M} \; M$$

$$\langle j, M' \,|\, J_+ \,|\, j, M \rangle = \sqrt{\frac{(j+M+1)(j-1)}{2}} \; \delta_{M'(M+1)}$$

$$\langle j, M' \,|\, J_- \,|\, j, M \rangle = \sqrt{\frac{(j+M)(j-M+1)}{2}} \; \delta_{M'(M+1)}$$

"SPIN-j REP" OF $SU(2)$

$$\left[J^{(j)}_{\underset{3,\pm}{a}} \right]_{k\ell} = \langle j, \overbrace{j+1-k}^{j-(k-1)} \,|\, J_a \,|\, j, \overbrace{j+1-\ell}^{j-(\ell-1)} \rangle$$

OR LABELING BY M VALUES

$$\boxed{\left[J^{(j)}_{a} \right]_{M'M} = \langle j, M' \,|\, J_a \,|\, j, M \rangle}$$

REPS: SPIN - $\frac{1}{2}$

$$J_1^{(1/2)} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{2}\sigma_1$$

$$J_2^{(1/2)} = \frac{1}{2}\begin{pmatrix} & -i \\ i & \end{pmatrix} = \frac{1}{2}\sigma_2$$

$$J_3^{(1/2)} = \frac{1}{2}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = \frac{1}{2}\sigma_3$$

STATES: $|\frac{1}{2}, \frac{1}{2}\rangle$
$|\frac{1}{2}, -\frac{1}{2}\rangle$

SPIN - 1:

$$\vec{J} = \frac{1}{\sqrt{2}}\left\{ \begin{pmatrix} & 1 & \\ 1 & 0 & 1 \\ & 1 & \end{pmatrix}, \begin{pmatrix} & -i & \\ i & 0 & -i \\ & i & \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix} \right\}$$

all traceless, hermitian matrices

states: $|1 \;\; 1\rangle$
$|1 \;\; 0\rangle$
$|1 \;\; -1\rangle$

THESE TWO ARE SPECIAL REPS

SPIN-$\frac{1}{2}$: DEFINING REP (FUNDAMENTAL)

↑ what we mean by "SU(2)"

SPIN-1: ADJOINT REP, EQUIVALENT TO A REPRESENTATION FURNISHED BY GENERATORS:

$$[d^{(Ad)}(T_a)]_{bc} = \overset{?}{-i}\, C_{abc}$$

$$[T_a, T_b] = i\, C_{abc} T_c$$

$$[d^{(Ad)}(T_a)]\, |T_b\rangle = |[T_a, T_b]\rangle = i C_{abc} |T_c\rangle$$

GENERATORS THEMSELVES ARE BASIS FOR REP

[BTW: $Tr(T_a T_b)$ IS A GOOD INNER PRODUCT ON $\mathcal{L}(G)$.]

OTHER REPS

SPIN-$3/2$: $|\frac{3}{2}, \frac{3}{2}\rangle, |\frac{3}{2}, \frac{1}{2}\rangle, |\frac{3}{2}, -\frac{1}{2}\rangle, |\frac{3}{2}, -\frac{3}{2}\rangle$

SPIN-2: $\underbrace{|2, 2\rangle, \cdots |2, -2\rangle}_{5 \text{ states}}$