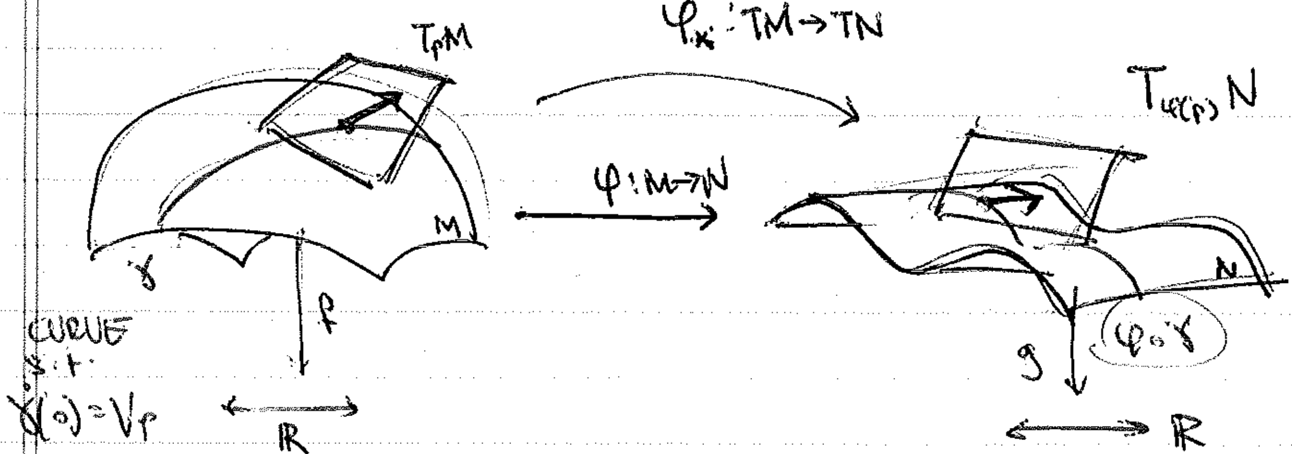
LEC 20: LIE ALGEBRAS = T_eG 11/7

QM: INTUITION - GENERATORS

RECALL: MAPS BETWEEN MANIFOLDS
DEFINE MAPS BETWEEN TANGENT BUNDLES



$\varphi \circ \gamma$: CURVE s.t. $\frac{d}{dt}(\varphi \circ \gamma)(0) = \varphi_* V_p$

$$\varphi_* : V_p \in T_pM \longrightarrow (\varphi_* V_p) \in T_{\varphi(p)}N$$

ACTS ON $f: M \to \mathbb{R}$ ACTS ON $g: N \to \mathbb{R}$

$$V_p(f) = \frac{d}{dt}(f \circ \gamma)_{t=0}$$

$$= \frac{\partial f}{\partial x^i} \underbrace{\frac{\partial \gamma^i}{\partial t}\Big|_{t=0}}_{\equiv V_p}$$

$\varphi_* V_p$ IS DEFINED SIMILARLY.

IN FACT, GIVEN THE INGREDIENTS ABOVE

(→ A CURVE γ that defines V_p)
(→ A MAP $\varphi : M \to N$)

$\varphi_* V_p$ DOES THE ONLY THING IT POSSIBLY CAN.

$$\underbrace{(\varphi_* V_p)}_{\in T_{\varphi(p)}N} \underset{N \to \mathbb{R}}{(g)} = \frac{d}{dt}\left(g \circ (\cdots)\right)$$

this has to be a PATH IN N SUCH THAT THE "TIME DERIVATIVE" @ $t=0$ GIVES $(\varphi_* V_p)$

how do we construct such an object using the tools available?

$\varphi \circ \gamma$ is a path in N

so define $\varphi_* V_p$ BY $\frac{d}{dt}(\varphi \circ \gamma)_{t=0}$

<u>then</u>:

$$(\varphi_* V_p)\, g = \frac{d}{dt}\left(\underbrace{g \circ \varphi}_{\text{a test function on } M \,\equiv\, \tilde{f}} \circ \gamma\right)_{t=0} = V_p \tilde{f}$$

SO NOW YOU KNOW HOW TO MAP VECTORS BETWEEN MANIFOLDS, GIVEN A MAP BETWEEN MANIFOLDS.

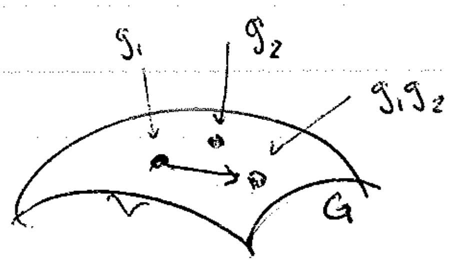
<u>BUT</u>: RECALL: LIE GROUP: GROUP THAT IS ALSO A MANIFOLD

$\underbrace{g \in G}_{\text{manifold}}$, BUT <u>ALSO</u> A MAP $\underbrace{g : G \times G \to G}_{\text{group}}$

eg. $U(1)$: $e^{i\theta}$ ← isomorphic to $SO(2)$

$$g_1 = e^{i\theta_1}$$
$$g_2 = e^{i\theta_2}$$

$$g_1 \cdot g_2 = e^{i(\theta_1+\theta_2)} \in G$$

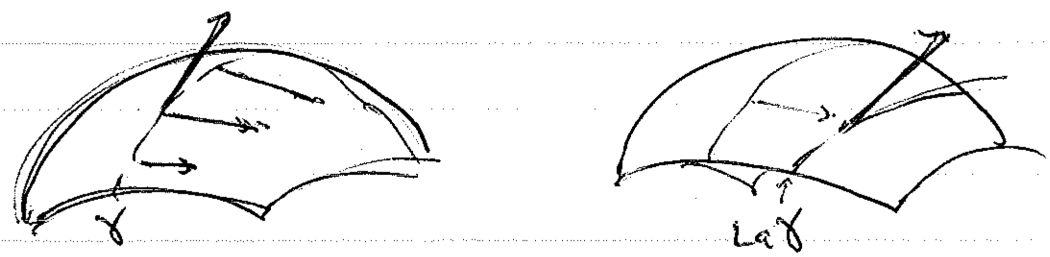


DEFINE A MAP: [LEFT] TRANSLATION BY a.
FOR $a, g \in G$, then $L_a : G \to G$ is

$$\boxed{L_a g = ag}$$

THIS IS A MAP "BETWEEN MANIFOLDS"
(except between manifold to itself)

↳ THEN FROM OUR WORK ABOVE, CAN MAP TANGENT VECTORS



DEF: A VEC. FIELD IS [LEFT]-INVARIANT IF

$$(L_a)_* V|_g = V|_{ag}$$

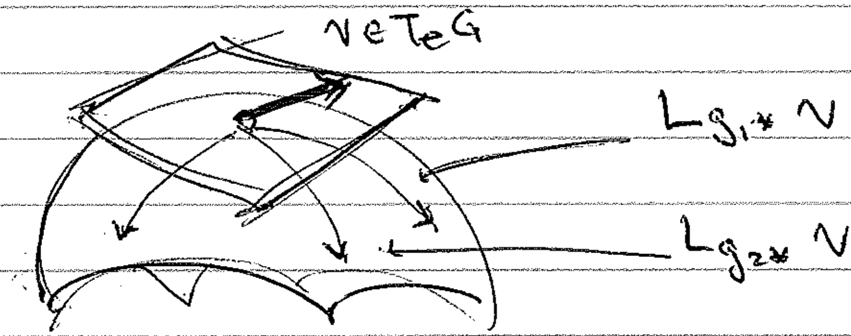
SO WHAT? THERE IS A SPECIAL ~~POINT~~ ELEMENT
OF EACH LIE GROUP: $\mathbb{1}$, THE IDENTITY.
(sometimes called e)

WE CAN CONSTRUCT [LEFT] INVT. VECTOR FIELDS
BY PUSHING ELEMENTS OF THE TANGENT SPACE
AT $\mathbb{1}$, $v \in T_{\mathbb{1}} G$

T_eG is EASIER TO READ

ie given $v \in T_{11}G$

then $\underbrace{V(v)|_g}_{\text{the vector @ } g \in G} = \underline{L_{g*}v}$ — PUSH v TO g



IN THIS WAY, CAN PUSH A COPY OF $v \in T_eG$ TO EVERY TANGENT SPACE, $T_gG \;\forall g \in G$.

ie MAPPED $T_eG \to TG$.

HW: SHOW THAT ~~LEFT INVT~~ VEC FIELD SO DEFINED IS INDEED [LEFT] INVARIANT:

$$(L_a)_* \underbrace{V|_g}_{(L_g)_* v} = \underbrace{V|_{ag}}_{(L_{ag})_* v}$$

SO: SHOW SOMETHING ABOUT THE COMPOSITIONS OF PUSH FORWARDS.

SO WE HAVE A MAP: $T_eG \to TG$

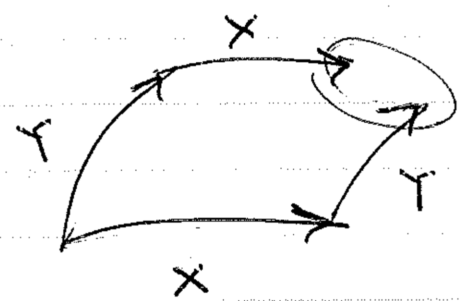
IN FACT, THIS MAP IS INVERTIBLE

SO WE CAN MAP $TG \to T_eG$.

↳ looks like we only have to deal w/ <u>one</u> tangent space.

<u>IMPORTANT CHECK</u>: WHAT ABOUT <u>LIE BRACKET</u>?

recall:



LIE BRACKET MEASURED NONCOMMUTATIVITY OF FLOWS.

↳ compare to rotating book example

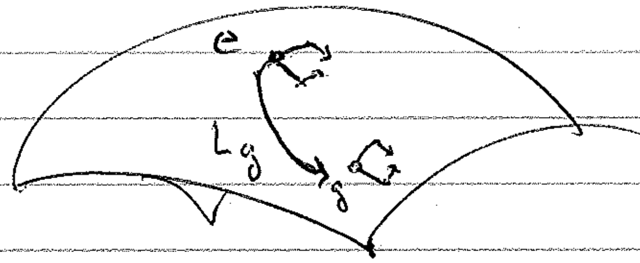
THIS IS PRECISELY THE IMPORTANT STRUCTURE OF A GROUP!

↳ when do symmetries "interact" w/ each other?

TURNS OUT (HW), $[X,Y]$ IS LEFT-INVARIANT:

$$(L_a)_*[X,Y]|_g = [X,Y]_{ag} \quad ✓$$

WHAT THIS MEANS:



the noncommutivity @ g is the same as @ e

⇝ T_eG seems to tell us everything.

DEF: THE LIE ALGEBRA of G is T_eG ($\mathcal{L}(G)$).
IT IS DEFINED BY THE LIE BRACKET, OR COMMUTATOR: $[\cdot,\cdot]$

PROPERTIES:
1) $[v,w] = -[w,v]$ (ANTISYM)
2) $[\alpha v_1 + \beta v_2, w] = \alpha[v_1,w] + \beta[v_2,w]$ (LIN)
3) BIANCHI IDENTITY:
~~$[v,[w,z]]$~~
$[[v,w],z] + [[z,v],w] + [[w,z],w] = 0$

⇝ LIE ALGEBRA IS A VECTOR SPACE w/ $[\cdot,\cdot]$.

DEF: ~~THE~~ THE BASIS ELEMENTS OF $\mathcal{L}(G) = T_e G$ ARE CALLED <u>GENERATORS</u>, T_i.

$$[T_i, T_j] = C_{ij}{}^k T_k \leftarrow \text{LINEARITY}$$

STRUCTURE CONSTANTS OF $\mathcal{L}(G)$

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eg <u>OUR OLD FRIEND $SO(2)$</u>

$$g(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$e = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} = g\big|_{\theta=0}$$

A CURVE PASSING THROUGH THE ORIGIN IS $\theta(t) = t$. $\to g(\theta(t))$ g'

$$\text{then } \frac{d}{dt} g(\theta(t))\Big|_0 = \begin{pmatrix} -\sin\theta & \cos\theta \\ -\cos\theta & -\sin\theta \end{pmatrix}\Big|_0 \dot{\theta}$$

$$= \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{\in \mathcal{L}(SO(2))}$$

GENERATOR OF ROT.

<u>MORE GENERALLY</u> eg $O(n)$

ORTHOG (real) $n \times n$ matrices

SOME PARAMETER FOR A CURVE IN G

$$M(t)\, M(t)^T = \mathbb{1}_{n \times n}$$

WHAT DOES TANGENT SPACE LOOK LIKE?

$$\dot{M} M^T + M \dot{M}^T = 0$$

EVALUATE @ $t=0$, WHERE $M(0) = \mathbb{1}$
BY CHOICE OF PARAMETERIZATION

$$\rightarrow \quad \left(dM/dt\right)_0 + \left(dM/dt\right)_0^T = 0$$

$$\Rightarrow \quad dM/dt\,\big|_0 \text{ is } \underline{\text{ANTISYMMETRIC}},$$

DIMENSIONALITY = $\frac{1}{2} n (n-1)$.

HW: do this for $SU(N)$

$so(2)$: $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$

$so(3)$: $\begin{pmatrix} 0 & & 1 \\ & 0 & \\ -1 & & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & & \\ & 0 & 1 \\ & -1 & 0 \end{pmatrix}$

FROM ALGEBRA TO GROUP:

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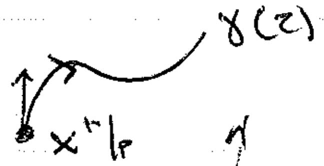
<u>RECALL</u>: WHEN WE DID LIE DERIVATIVES (LEC 18) FLOW ALONG AN INTEGRAL CURVE



$$e^{\Delta\tau \frac{d}{d\tau}} X^\mu|_p$$

exponentiation of a tangent vector.

integral curve of a vector field

LET $\sigma : \mathbb{R} \to G$ BE A <u>1-PARAMETER SUBGROUP</u>; INTEGRAL CURVE OF LEFT-INVT. VEC. FIELD, V

$$\sigma(0) = e \quad ; \quad \sigma(s)\sigma(t) = \sigma(s+t)$$

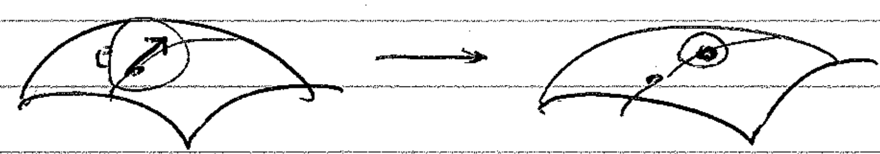
$$\boxed{V_{\sigma(t)} = L_{\sigma(t)*} V}$$

σ IS THE INTEGRAL CURVE FROM PUSHING $V \in T_eG$

def EXPONENTIAL MAP $\exp : \underbrace{T_e G}_{\mathcal{L}(G)} \to G$

$$\exp(v) = \sigma_v(1)$$ one particular elem of G

1 PARAM SUBGROUP GENERATED BY $V(v)$



HOW TO MAP TO OTHER ELEMENTS?

$$e^{tv} = \sigma_v(t)$$

$t \in \mathbb{R}$, $v \in T_e G$

$$\text{s.t. } e((t_1 + t_2)v) = e^{t_1 v} e^{t_2 v}$$