MY NOTES
follow Gutowski

LEC 19: GROUP THEORY overview NOV 4

USEFUL REFERENCES
- CAHN Semi-Simple Lie Algebras & their Reps
- JONES Groups, Reps, & Physics
- TUNG Group Theory in Physics
- GEORGI Lie Algebras in Particle Physics

Group theory ⟷ symmetries

ABSTRACT, MATHEMATICAL OBJECTS

$\boxed{\text{Representation Theory}}$

matrices / diff. ops acting on physical quantities like wavefunctions

1. FINITE: fe # of transformations
   eg like symmetries of polyhedra

2. CONTINUOUS ~~INFINITE~~ ⟵ LIE GROUP: ∞ # symmetries
   re. ~~cont~~ ∃ PARAMETERS

⇑
we'll focus on this

GROUP: A SET G WITH A MAP (MULTIPLICATION) (MULT): $G \times G \to G$ THAT SATISFIES, $\forall g \in G$:

1. $\exists \mathbb{1} \in G$ w/ $\mathbb{1} g = g \mathbb{1} = g$

2. $\exists g^{-1} \in G$ w/ $g g^{-1} = g^{-1} g = \mathbb{1}$

3. $g_1(g_2 g_3) = (g_1 g_2) g_3$

LIE GROUP: A GROUP, G, THAT IS ALSO A SMOOTH, DIFFERENTIABLE MANIFOLD. MULTIPLICATION & INVERSE ARE SMOOTH.

MOST INTERESTING LIE GROUPS ARE MATRIX GROUPS IN VARIOUS DIMENSIONS

$$\hookrightarrow \underset{\sim}{g}^{ij}(x^{\alpha})$$

PARAMETERS

not metric!!

$$\text{eg} \quad SO(2) \Rightarrow \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

SPECIAL ORTHOG 2x2 MATRICES

θ: one param.

$M^T M = \mathbb{1}$

det $M = 1$

<u>MORE GENERAL</u>:

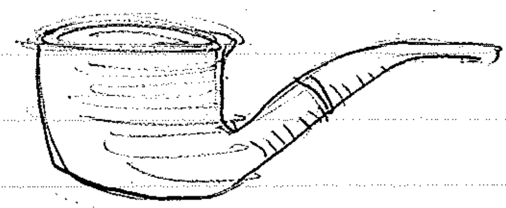
$$GL(n,\mathbb{C}) \ni \begin{pmatrix} z_{11} & z_{12} & \cdots \\ z_{21} & \ddots & \\ \vdots & & \end{pmatrix}$$

INVERTIBLE

GENERAL LINEAR GROUP; $n \times n$ MATRICES w/ $\mathbb{C}$ ELEMENTS.

→ n^2 parameters.

Representations



RENÉ MAGRITTE
THE TREACHERY of IMAGES

"ceci n'est pas une pipe"
[implied: this is a representation of a pipe]

<u>DEF</u>. LET V BE A FINITE DIM VECTOR SPACE
state (ket) space
LET $GL(V)$ BE THE SPACE OF LINEAR TRANS: $V \to V$

A <u>REPRESENTATION</u> OF A ~~LIE~~ GROUP G ACTING ON V IS A MAP $D: G \to GL(V)$

SUCH THAT $\mathcal{D}(g_1 g_2) = \mathcal{D}(g_1)\mathcal{D}(g_2)$
$\forall g_1, g_2 \in G$.

$\mathcal{D}$ IS A HOMOMORPHISM

THE DIMENSION OF THE REPRESENTATION IS $\dim \mathcal{D} = \dim V$

eg. IF $\mathcal{D}$ IS A REP OF G ON V,

$$\mathcal{D}(\mathbb{1}) = \mathbb{1}$$

abstract element

unit matrix in V

nb. FINITE GROUPS

eg. $\mathcal{D}(g^{-1}) = [\mathcal{D}(g)]^{-1}$

THE TRIVIAL REPRESENTATION:

$$\mathcal{D}(g) = 1 \quad \forall g \in G$$

$\rightarrow$ satisfies all rules!

n-DIMENSIONAL TRIVIAL REP:

$$\mathcal{D}(g) = \mathbb{1}_{n\times n} \quad \forall g \in G.$$

not faithful (injective)

all of our examples will be

IF G IS A MATRIX LIE GROUP $\subset GL(n, \mathbb{R}/\mathbb{C})$,
then the elements themselves act on
n-component vectors.

FUNDAMENTAL REP. $D(g) = g$

eg $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ for $SO(2)$

OTHER REPS OF $SO(2)$

$$\left(\begin{array}{cc|c} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{c|c} e^{i\theta} & 0 \\ \hline 0 & e^{-i\theta} \end{array}\right)$$

REDUCIBLE REPRESENTATIONS

$e^{i\theta}$

in fact: FUNDAMENTAL OF $U(1)$

1×1 UNITARY MATRICES
$U^\dagger U = 1.$

$U(1) = SO(2)$

A LIE GROUP G IS COMPACT IF G IS <u>COMPACT</u> AS A MANIFOLD.

$\longleftrightarrow$ CLOSED & BOUNDED

contains limit points

eg. $SU(n)$: SPECIAL UNITARY $n\times n$ MATRICES

$\det M = 1$ $\quad$ $M^\dagger M = \mathbb{1}_n$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} = \begin{pmatrix} |a|^2+|b|^2 & ac^*+bd^* \\ ca^*+db^* & |c|^2+|d|^2 \end{pmatrix}$$

$$\equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

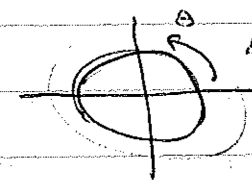
so elements can't be larger than 1 in modulus.

eg. $SO(1,1)$ $\leftarrow$ 2D LORENTZ GROUP

$$\begin{pmatrix} \cosh R & \sinh R \\ \sinh R & \cosh R \end{pmatrix}$$ all unbounded

<u>DEF</u>. G IS <u>CONNECTED</u> IF ANY 2 POINTS IN THE GROUP CAN BE LINKED BY A CONTINUOUS CURVE IN G.

eg. $SO(2) = S^2 \leftrightarrow$ connected



every element in $SO(2)$ can be mapped to S^2

eg $O(N)$ IS NOT CONNECTED

(ORTHOG $N \times N$ MATRICES, $M^T M = \mathbb{1}_n$

OBSERVE $\det(M^T M) = [\det(M)]^2 = 1$

$\Rightarrow \det M = \pm 1$

CONSIDER ~~THE MATRICES~~ AN $M \in O(N)$

w/ $\det M = -1$. ↳ eg $\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \in O(2)$

$\gamma : [0,1] \to O(N)$

THEN IF CONNECTED, $\exists$ <u>PATH</u> γ FROM $\mathbb{1}$ TO M

CONTINUOUS $\gamma(0) = \mathbb{1}$, $\gamma(1) = M$

↳ $\det \gamma(t)$ IS $\begin{cases} \text{continuous} \\ \pm \end{cases}$

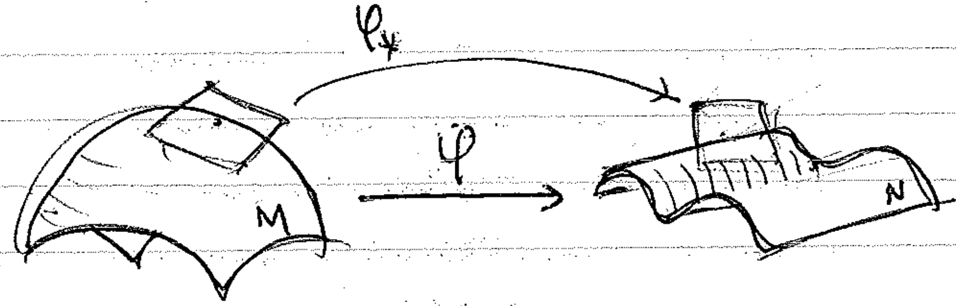
↳ inconsistent.

A LITTLE MORE GEOMETRY

LAST TIME: VECTOR FIELD $\Rightarrow$ MAP : $M \to M$
"velocity field flow"

NOW: MAP: $M \to N \Rightarrow$ MAP $TM \to TN$

tangent BUNDLES
$\{T_pM\}$ for all $p \in M$



φ_*: "PUSH FORWARD"

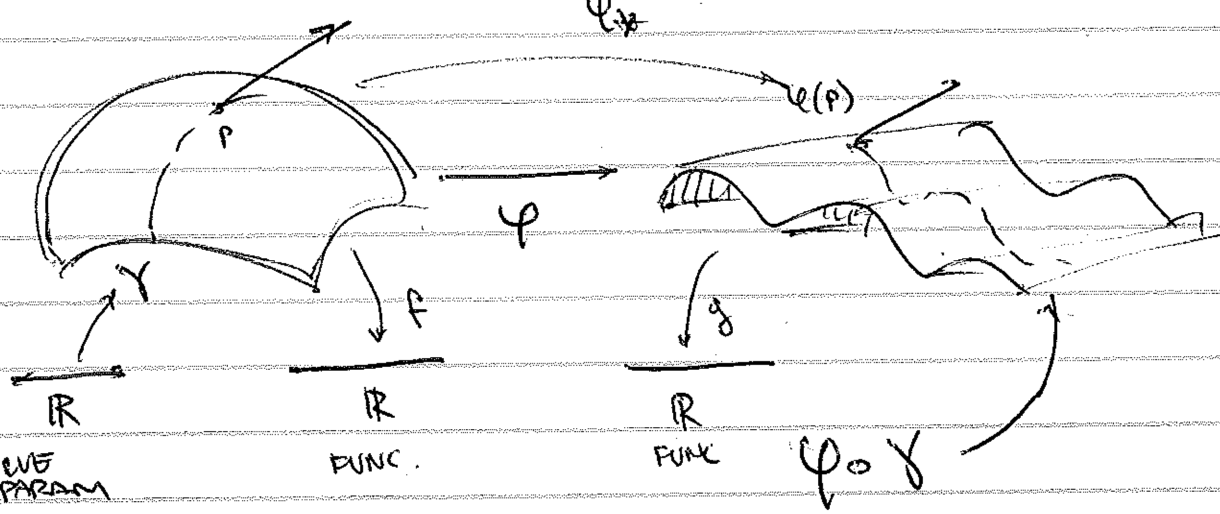
LET $\gamma : \mathbb{R} \to M$ BE A CURVE IN M
WITH $\gamma(0) = p \in M$

LET $V_p \in T_pM$ BE TANGENT VEC OF γ @ p

directional derivative acting on
test functions f
$V_p(f) \leftarrow (d_pf(V_p))$ IS A #

$\frac{d}{dt}(f \circ \gamma)$

$\boxed{V_p = \frac{d}{dt}\gamma|_0}$



V_p ACTS AS DIR. DERIV. ON f VIA

$$V_p f = \frac{d}{dt}(f\circ\gamma)$$

vec. fun. test func. curve

$\varphi_* V_p \in T_{\varphi(p)}M$ ACTS AS DIR DERIV ON g VIA

$$(\varphi_* V_p)\, g = \frac{d}{dt}\Big(g\circ\underbrace{(\varphi\circ\gamma)}_{\text{CURVE IN N}}\Big)$$

test func.

$$= \frac{d}{dt}\big((g\circ\varphi)\circ\gamma\big)$$

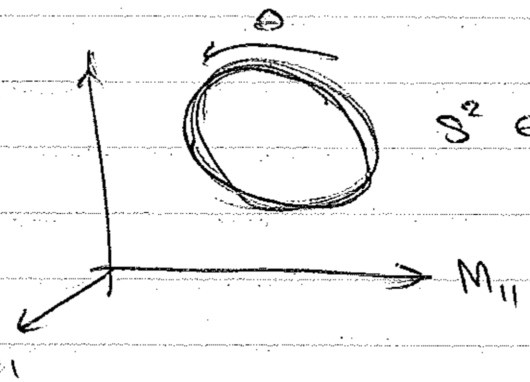
test fcn. $M\to\mathbb{R}$ CURVE IN M

$$= V_p(g\circ\varphi)$$

Why is this important?

LIE GROUPS ARE GROUPS THAT ARE ALSO MANIFOLDS

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \longleftarrow \underbrace{\begin{pmatrix} M_{11}(\theta) & M_{12}(\theta) \\ M_{21}(\theta) & M_{22}(\theta) \end{pmatrix}}_{\text{4D SPACE}}$$



$S^2 \in \sim\mathbb{R}^4$

(WHAT ARE TANGENT VECTORS?)

AS GROUPS, THEY HAVE A GROUP MULTIPLICATION DEFINED. AS MANIFOLDS, THIS GROUP MULTIPLICATION IS A MAP: $M \to M$.

or transformation

GIVES A WAY TO MOVE TANGENT VECTORS AROUND.

$$\frac{d}{dt}\begin{pmatrix} c_t & s_t \\ -s_t & c_t \end{pmatrix}\Big|_{t=0} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

LEFT-INVARIANT VECTOR FIELDS

LET $a, g \in G$.

<u>DEF.</u> LEFT TRANSLATION: $\boxed{\begin{array}{l} L_a : G \to G \\ L_a g = ag \end{array}}$ by a

L_a IS A MAP FROM $G \to G$

CAN DEFINE <u>PUSH FORWARD</u> OF VECTORS $V \in T_g G$

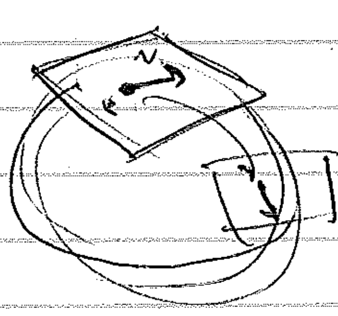
$L_{a*}(V|_g)$

<u>DEF</u>: A <u>LEFT-INVARIANT</u> VECTOR FIELD, $X \in T(G)$ IS ONE SUCH THAT

$$\boxed{L_{a*}(X|_g) = X|_{ag}}$$

vector @ origin

$\forall\, v \in T_e G$, CAN CONSTRUCT A <u>UNIQUE</u> LEFT-INVT. <u>VECTOR FIELD</u> $X(v) \in TG$ BY PUSHING IT:



$$\boxed{X(v)|_g = L_{g*}\, v}$$

~~pf: $X(v)|_{ag} = L_{(ag)*} v$~~