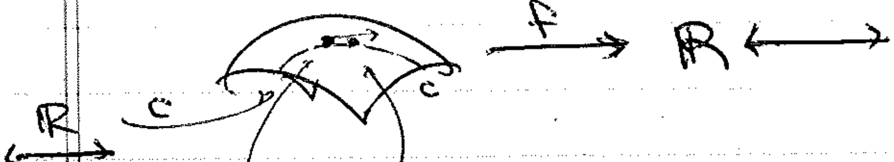
LEC 18: MOVING TENSORS 1 NOV

Refs: Stogo + Schutz Geom.

- HW6 - MANY CORRECTIONS / HINTS / PROBS AFTER 2.1 NOW OPTIONAL

Q: What is the "derivative" of a tensor?

FUNCTIONS ARE EASY!



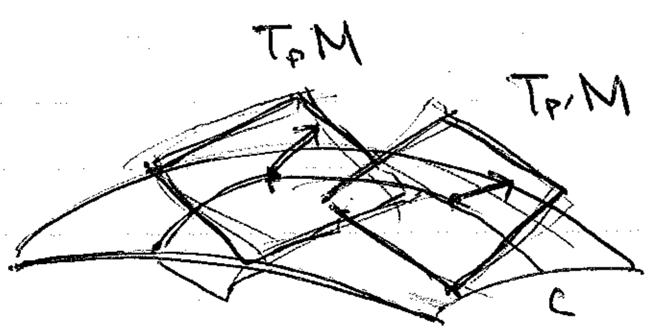
$f(x)$ $f(x+\Delta x)$

$x+\Delta x = x^i + \underbrace{\frac{\partial x^i}{\partial t} dt}_{\text{ALONG SOME PATH } c} + \dots$

USUAL SENSE OF DIRECTIONAL DERIVATIVE

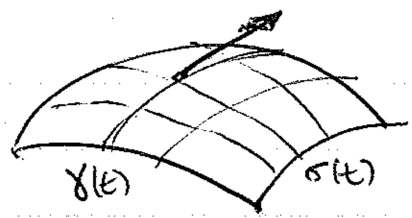
MORE COMPLICATED TENSORS ARE SUBTLE

eg a VECTOR LIVES ON A TANGENT SPACE.



A VECTOR FIELD IS A MAP FROM $M \to TM$ BUNDLE.

NEED TO COMPARE VECTORS IN DIFFERENT TANGENT SPACES



USE: INTEGRAL CURVES
EACH CURVE: HAS SOME "TIME PARAMETER"

such that if $\underline{\gamma}(t_0) = p \in M$, then $\dot{\underline{\gamma}}(t_0)$ is A TANGENT VECTOR $\in T_pM$

other tangent vectors in T_pM come from the "velocities" of other trajectories, say $\sigma(t)$

MORE IMPORTANTLY: $\gamma(t_0+\Delta t)$ is a point $p' \in M$ NEAR p. $\dot{\gamma}(t_0+\Delta t)$ is a tangent vector in $T_{p'}M$.

NB: WE'RE ADDRESSING THE PROBLEM OF: HOW TO COMPARE WHETHER VECTORS IN T_pM & $T_{p'}M$ ARE PARALLEL. ~~IN GEN,~~ WE DON'T HAVE THE MACHINERY TO DO THAT! (HAVING A METRIC WILL DO IT) $\longrightarrow$ THE STRUCTURE TO DO THIS IS CALLED A CONNECTION $\leftrightarrow$ COVARIANT DERIVATIVE

$\hookrightarrow$ WE WILL DEF A DIFFERENT DERIVATIVE THAT IS ALSO IMPORTANT.

VECTOR FIELD: $V(x) = V^\mu(x)\, \partial/\partial x^\mu$

FOR NICE ENOUGH V,
CAN ALWAYS WRITE $V^\mu(x)$ AS ~~INTEGR~~
TANGENT VECTOR OF AN INTEGRAL CURVE

$$\boxed{\frac{dx^\mu(\tau)}{d\tau} = V^\mu(x)}$$ ← set of 1st O DE, SOLUTION EXISTS

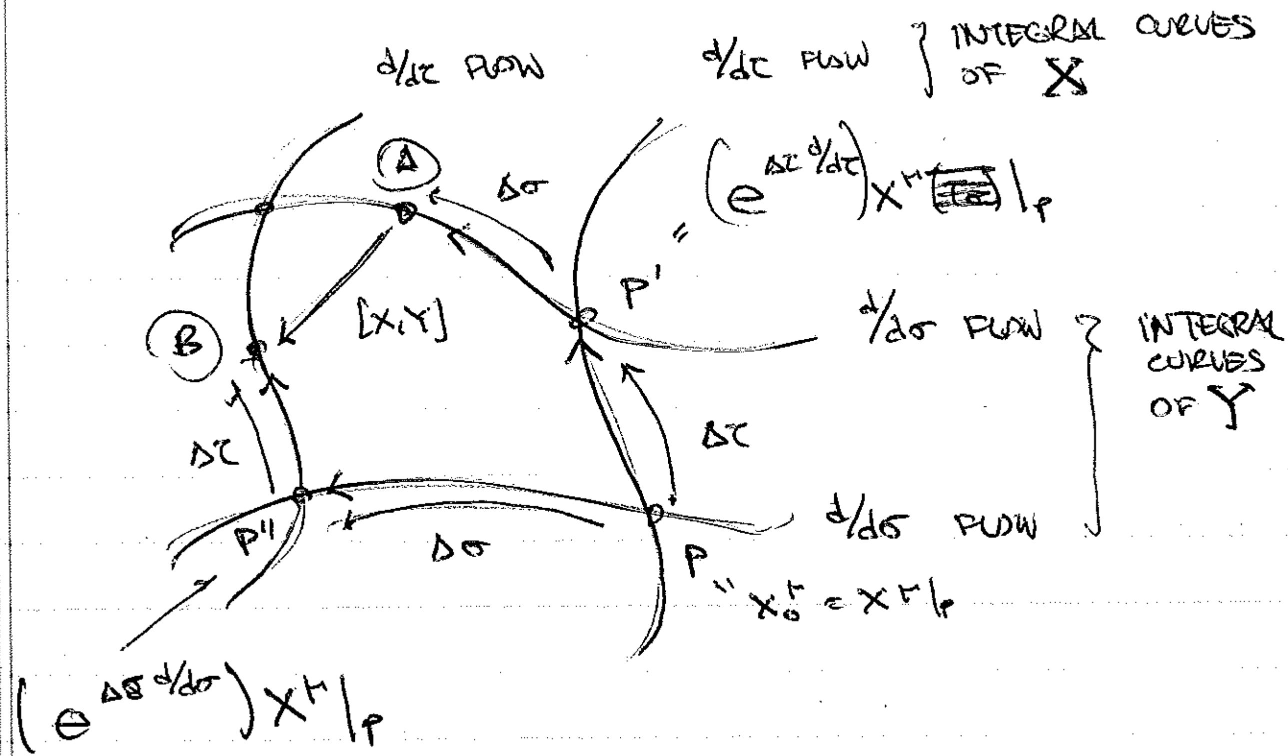
INTUITION FROM QM: $\frac{d}{d\tau}$ IS AN INFINITESIMAL TRANSLATION OPERATOR, LIKE MOMENTUM:

$$x^\mu(\tau_0 + \Delta\tau) = x^\mu(\tau_0) + \Delta\tau \frac{dx^\mu(\tau_0)}{d\tau} + \ldots$$

$$= \left(1 + \Delta\tau \frac{d}{d\tau} + \frac{1}{2}\Delta\tau^2 \frac{d^2}{d\tau^2}\right) x^\mu \Big|_{\tau_0}$$

$$= \boxed{e^{\Delta\tau (d/d\tau)}}\, x^\mu \Big|_{\lambda_0}$$

EXPONENTIATION: FINITE TRANSLATION.



<u>then</u>: $$A = e^{\Delta\sigma \frac{d}{d\sigma}}\, e^{\Delta\tau \frac{d}{d\tau}}\, X^\mu \Big|_p$$

$$B = e^{\Delta\tau \frac{d}{d\tau}}\, e^{\Delta\sigma \frac{d}{d\sigma}}\, X^\mu \Big|_p$$

$$X^\mu(A) - X^\mu(B) = \underbrace{\left[e^{\Delta\sigma \frac{d}{d\sigma}}, e^{\Delta\tau \frac{d}{d\tau}} \right]} X^\mu \Big|_p$$

$$= \left[1 + \Delta\sigma \frac{d}{d\sigma} + O(\Delta\sigma^2),\ 1 + \Delta\tau \frac{d}{d\tau} + O(\Delta\tau^2) \right]$$

$$= \Delta\sigma \Delta\tau \left[\frac{d}{d\sigma}, \frac{d}{d\tau} \right] X^\mu \Big|_p$$

$$= \Delta\sigma \Delta\tau \left(\frac{d}{d\sigma} X^\mu - \frac{d}{d\tau} Y^\mu \right) \Big|_p$$

$$\equiv \Delta\sigma \Delta\tau \underbrace{[X, Y]^\mu}_{\text{LIE BRACKET.}}$$

COMMUTATOR OF TANGENT VEC

CAN THINK OF THIS AS COMMUTATORS
OF OPERATORS (IN QM SENSE!)

$$XY = X^\mu \partial_\mu (Y^\nu \partial_\nu) = \underbrace{X^\mu Y^\nu \partial_\mu \partial_\nu}_{\text{WHAT IS THIS?!}} + X^\mu (\partial_\mu Y^\nu) \partial$$

ACTING ON TEST FUNCTION,
GIVES 2ND DERIVATIVE.
THIS DOES NOT TRANSFORM
NICELY.

$$\text{BUT } [X,Y] = \underbrace{\left[X^\mu (\partial_\mu Y^\nu) - Y^\mu (\partial_\mu X^\nu) \right]}_{[X,Y]^\mu} \partial_\nu$$

GIVES A NEW VECTOR FIELD;
bona fide directional derivative

CAN ALSO APPLY TO MORE COMPLICATED TENSORS
... REQUIRES MORE WORK

COORDINATES VS. "JUST" INTEGRAL CURVES

GIVEN A SET OF INDEP. VECTOR FIELDS $X, Y, Z, \ldots$ on TM, WHEN DO THEIR INTEGRAL CURVES FORM COORDINATES FOR M?

↳ when is $\{X, Y, Z, \ldots\}$ INTEGRABLE?

SUFFICIENT: $[X, Y] = 0$, ... etc.

then X integ. curves have const. Y integ curve coordinates

When do integ. curves FOLIATE M

NECESSARY: integrable when:

$$[X_{(I)} X_{(J)}] = \underbrace{C_{IJ}{}^{K}}_{\text{some function on } M} X_{(K)} \qquad (\text{"involutive set"})$$

in fact (FROBENIUS THM): INVOLUTIVE ⇔ INTEGRABLE

<u>LIE DERIVATIVE</u> : takes tensors to same type

$\mathcal{L}_X f = Xf$ as usual

$\mathcal{L}_X Y = [X, Y]$

$\mathcal{L}_X$ defined to satisfy LEIBNIZ RULE

$\omega = \omega_\mu(x)\, dx^\mu$

$V = V^\nu(x)\, \partial/\partial x^\nu$

eg. $\mathcal{L}_X \omega$?

(FUNCTION GIVEN A VECTOR FIELD

LET V BE ARB. V. FIELD

DIR. DERIV.

$$\mathcal{L}_X \underbrace{\omega(V)}_{\text{func.}} = X[\omega(V)]$$

$$= \underbrace{(\mathcal{L}_X \omega)}_{\text{WANT THIS}} V + \omega(\underbrace{\mathcal{L}_X V}_{[X,V]})$$

$$\Rightarrow (\mathcal{L}_X \omega)\cancel{V} = X[\omega(V)] - \omega[X, V]$$

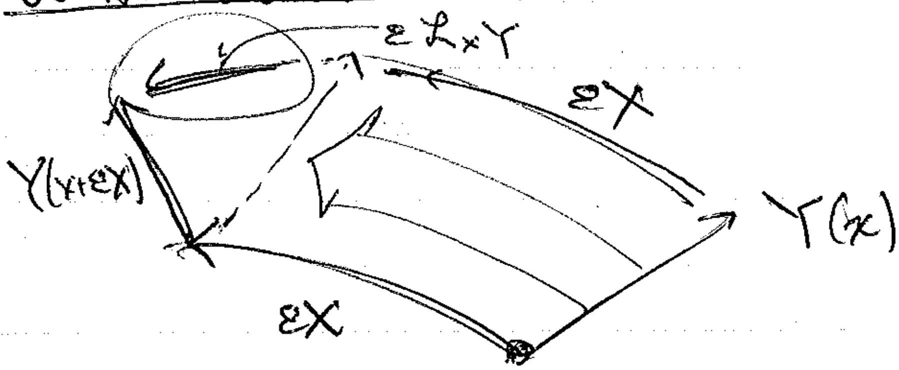
$$\cancel{= (X^\nu \partial_\nu (\omega)_\mu - \omega_\nu \partial}$$

$$\cancel{X^\nu [\partial_\nu \omega}$$

WHICH YOU CAN WRITE IN COMPONENTS

$$= (X^\nu (\partial_\nu \omega_\mu) + \omega_\nu (\partial_\mu X^\nu))\cancel{V^\mu}$$

WHAT LIE DERIV DOES



MOST NATURAL DERIVATIVE:

$$\text{"}Y(x+\varepsilon X) - \underline{Y(x)}\text{"}$$

BUT HAVE TO DRAG THIS TO $x+\varepsilon X$

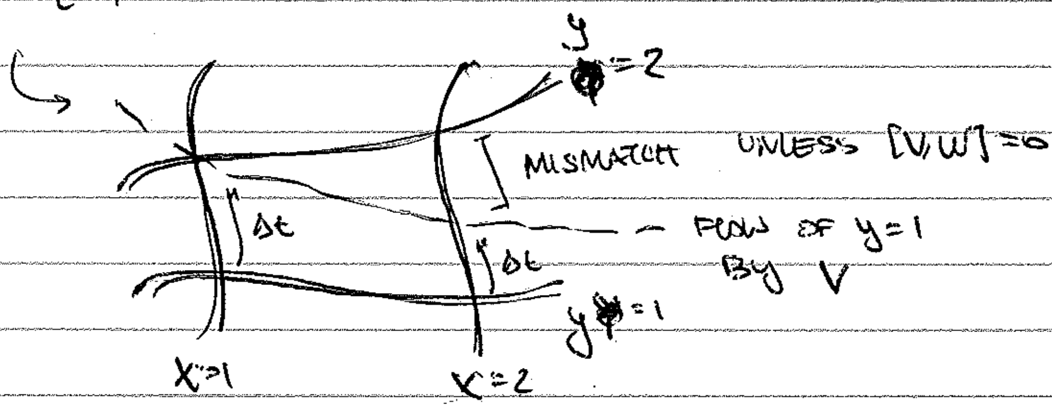
ONE APPLICATION: ISOMETRIES :
FLOW s.t. $\mathcal{L}_X g = 0$, METRIC IS CONSTANT.
$\longrightarrow$ KILLING FIELDS ; SYMM. OF SPACETIME

REMARKS

- COORDINATE VS. NON COORDINATE

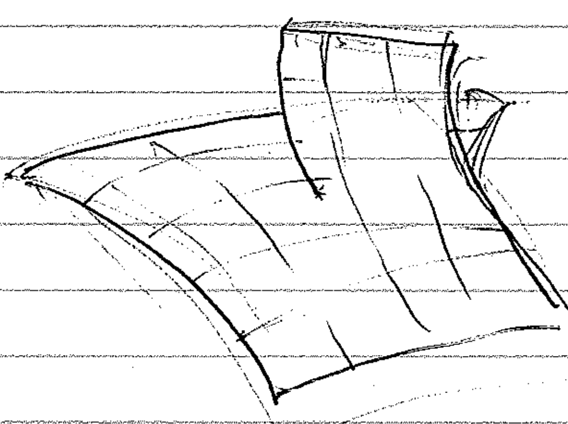
for 2D manifold, suppose 2 vec fields
V & W w/ integral curves x & y

WE SAID: $\{x\}$ & $\{y\}$ ARE COORDINATES
IF $[V, W] = 0$



ie: THE INTEGRAL CURVE $y=2$ OF W
IS NOT A CURVE OF CONSTANT t (V flow)

eg. 2 SETS OF INTEGRAL CURVES IN
3 SPACE CAN
FORM A MESH IN
SOME PLACES
BUT NOT OTHERS
→ NOT A 2D SUBSPACE



LIE BRACKET SHOULD LOOK FAMILIAR. CONSIDER

$$\frac{\partial}{\partial \phi} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

$|e_\phi\rangle$, L_z SIMILAR FOR L_y, L_x

then LIE BRACKETS ARE

$$[L_x, L_y] = -L_z$$
$$[L_y, L_z] = -L_x$$
$$[L_z, L_x] = -L_y$$

& cyclic

LOOKS LIKE 3D VECTOR BUNDLE

BUT IT'S ACTUALLY <u>DEGENERATE</u>
EASY TO SEE: INTUITIVELY, WE KNOW THESE TRANSFORMATIONS GENERATE ROTATIONS, PRESERVE r

$r = \sqrt{x^2+y^2+z^2}$; dr ACTS ON VECTORS

& $dr(L_i) = 0$.

So: $L_{x,y,z}$ ALL TANGENT TO r = CONST SPHERE.
& they <u>GENERATE</u> THIS SPHERE.

SPEAKING OF CONSTRAINTS :

ALSO: INVOLUTIVE $\Leftrightarrow [X_{(i)}, X_{(j)}] = C_{ij}{}^{k} X_{(k)}$

$\updownarrow$

holonomic constraints in mechanics

$\uparrow$ state is path independent

DOESN'T COME FROM A POTENTIAL

eg: RESTRICTING MOTION TO A SURFACE, WE JUST SAW, CAN BE WRITTEN AS

$$\omega(q)[\dot{q}] = \omega_\mu(q)\, dq^\mu [\dot{q}] = 0$$

ACT ON

(recall: Mechanics: PHASE SPACE)

$(q, \dot{q})$

INTEGRABLE $\leftrightarrow$ INVOLUTIVE (nice)

~~in general, ... constraints~~ :

eg: $\omega = x\,dx + y\,dy + z\,dz = r\,dr$

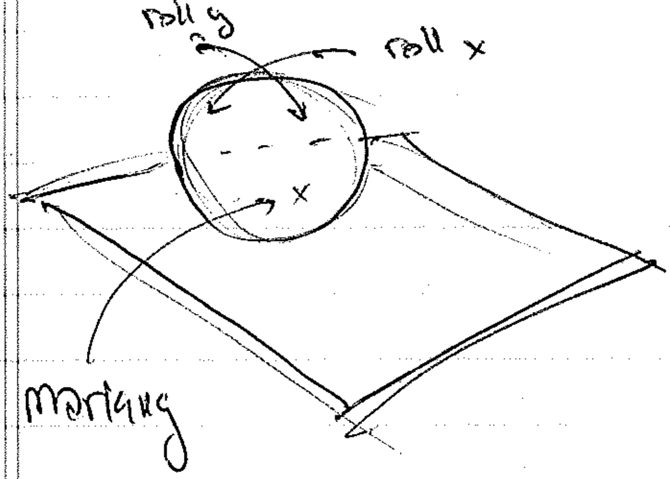
$\omega(\dot{q}) = 0 \leftrightarrow$ constrained to sphere

$\hookrightarrow$ eg. constraining to const E surface.

$\rightarrow$ holonomic. motion restricted to surface.

→ PARKING
→ ROLLING BALL / COIN w/ no SLIP
→ FALLING CATS
↳ SWIMMING ANIMALS

NONHOLONOMIC



5D CONFIG SPACE

$\mathbb{R}^2 \times S^3$

cm ↑ ↑ EULER ∡'s

No slip: 2 CONDITIONS

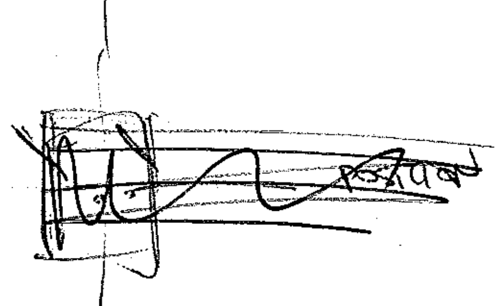
AXIAL ROT DOESN'T "DO MUCH"

→ involution of ROLLS?

BY taking LIE BRACKETS, WE ~~can~~ END UP w/ FIVE INDEP. VEL. VECTOR FIELDS

↳ roll x & roll y NOT INVOLUTION.

SO GIVEN A CONFIG, ∃ PATH IN PHASE SPACE TO GO TO ANY OTHER BY FLOWING ALONG ROLL y & ROLL x INTEGRAL CURVES.



2 ↳ PARKING: