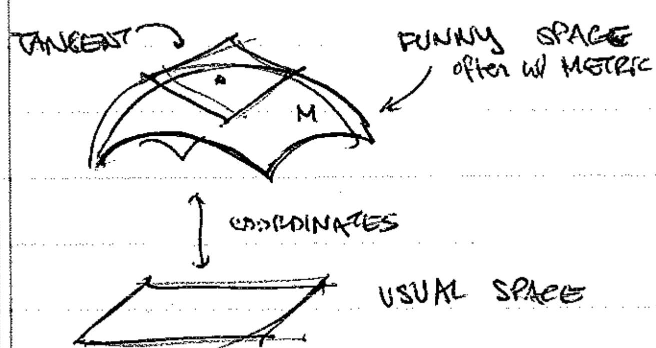
LEC 17 : POTENTIAL THEORY 31 OCT

WHY SO MUCH EM+SR: THIS IS BASIC "CULTURAL" GROUNDING FOR ALL PHYSICS

STORY THUS FAR



"coordinate free"

CALCULUS ON THIS SPACE.

d OPERATOR

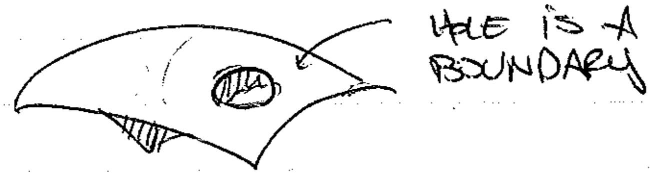
differential forms ω

antisym tensors that generalize volume

WHAT IS HINTED (beyond scope of this class):

→ CALCULUS ON CURVED / FUNNY SPACES CAN GET WEIRD!

- CURVED SPACES → GR
- SOMETIMES SPACES HAVE HOLES!



STOKES' THM PICKS UP THIS BOUNDARY ...

SO ALL THIS MACHINERY IS GREAT FOR E&M IN CURVED SPACE. (ALSO GEOMETRIC MECHANICS ON CONSTRAINED SYSTEMS!)

BUT THIS MACHINERY IS ALSO USEFUL IN ORDINARY FLAT SPACE(TIME).

CHECK (HW)

[EXPT.] $\Rightarrow$ $\vec{\nabla} \times \vec{E} + \dot{B} = 0$, $\vec{\nabla} \cdot \vec{B} = 0$ $\Longleftrightarrow$ $\boxed{dF = 0}$

POINCARE LEMMA: FOR A SMOOTH, CONTRACTIBLE (NICE) MANIFOLD [no holes!],

$$\underbrace{d\underbrace{\omega}_{k\text{ form}}}_{k+1\text{ FORM}} = 0 \Longleftrightarrow \omega = d\underbrace{\rho}_{k-1\text{ FORM}}$$

"closed form" "exact form" COMES FROM A POTENTIAL, ρ

POINCARE says: for suff. nice space, all closed forms are exact.

(Alternatively: closed forms are locally exact!

EXACT $\Rightarrow$ CLOSED is THE EASY pf SINCE $d(d\omega) \equiv 0$.

So: $dF = 0 \iff F = dA$

ELECTROM. POT.

$A_\mu = (\varphi, \underline{A})$

the moment we say EM FIELDS ($F \supset \underline{E}, \underline{B}$) come from a potential, we get half of Maxwell's eqns. FROM GEOMETRY. FOR FREE

You'll spell this out in the HW!

WHAT ABOUT THE REST OF MAXWELL?

IN COMPONENTS

$$\begin{aligned} \vec{\nabla} \times \vec{B} - \dot{\vec{E}} &= 4\pi \vec{J} \\ \vec{\nabla} \cdot \vec{E} &= 4\pi \rho \end{aligned} \iff \partial_\nu F^{\mu\nu} = 4\pi J^\mu$$

BUT HOW TO WRITE THIS IN A COORDINATE-FREE WAY?

LHS IS NOT ddA ... RHS ~~is~~ IS NOT 0

... FURTHER, IT IS NOT A 3-FORM!

INTRODUCE: HODGE STAR $*$: takes forms to their "complement"

$*$: IF $\mathcal{M}$ HAS DIM. n & has a $\underline{k}$-form $(k \leq n)$, ω, then the Hodge star gives

$(*\omega)$, an $(n-k)$-form:

ACTION ON BASIS:

$$* dx^{i_1} \wedge \dots \wedge dx^{i_k} =$$

$$= \frac{1}{(n-k)!} \sqrt{\det g}\, \varepsilon_{j_1 \dots j_n}\, dx^{j_{k+1}} \wedge \dots \wedge dx^{j_n}$$

TO COMPENSATE FOR CONTRACTION

Jacobian

Levi-Civita

$(n-k)$-form BASIS

$=1$ FOR FLAT COORDS

$$\times \left(g^{i_1 j_1} \dots g^{i_k j_k} \right)$$

inverse metrics TO MATCH INDICES.

eg: $*dx = dy \wedge dz$ in 3D EUCL.

$*dx \wedge dy = +dz \wedge dt$ in 4D MINK.

SO WHAT: IN YOUR HW: YOU WILL CHECK THAT

$$*F = \tilde{F}_{\mu\nu}\, dx^\mu \wedge dx^\nu$$

$\tilde{F}_{\mu\nu}$ is $F_{\mu\nu}$ w/ $\underline{E} \leftrightarrow \underline{B}$

SO REMAINING PART OF MAXWELL'S EQNS SEEMS TO HAVE A $d(*F)$

eg dF HAD $\vec{\nabla}\cdot\vec{B}$, so
$d*F$ HAS $\vec{\nabla}\cdot\vec{E}$

further, $d*F = \underbrace{d * d}A$

NOT IDENTICALLY ZERO.

BUT: $d*F = d*dA$ IS A 3-FORM.

So MAXWELL'S REMAINING EQS: $d*F = 4\pi J$

BUT BY HODGE STAR, $*J$ IS A 1-FORM. ✓ $\Big\{$ J IS A 3-FORM IN 4D MINKOWSKI

nb: this eq DOES NOT COME FROM GEOMETRY; IT HAS TO COME FROM PHYSICS ...

IN PARTICULAR, $d\star F = 4\pi J$ COMES FROM A LEAST ACTION PRINCIPLE

↓ ← LAGR. DENSITY

$$S = \int dt\, L = \int \underbrace{d(\text{vol})}_{\text{VOL FORM ON MINKOWSKI}} \mathcal{L}$$

$\therefore$ this $\star$ OPERATOR IS REALLY GREAT FOR WRITING LAGRANGIANS: $\star$ is "all the form-ness you need to make a volume form"

SO $(F \wedge \star F)$ IS A GOOD LAGRANGIAN FOR E$\&$M IN ANY DIMENSION.

HEURISTICALLY:

COUPLING TO SOURCE ↓ ↓ ← CURRENT / SOURCE

$$\mathcal{L} = \underbrace{dA \wedge \star d A}_{\text{DERIVATIVE TERMS} \to \text{DYNAMICS OF } A} + 4\pi A \wedge J$$

$$\sim -A \wedge d\star dA + 4\pi A \wedge J$$

$$\frac{\partial \mathcal{L}}{\partial A} = \boxed{-d\star dA + 4\pi J = 0}$$

notes: ① now you can do E$\&$M in any dim.

② this template follows for any force (GRAVITY too, but it's a little more subtle)

MORE NOTES

3. IF $J=0$, then MAXWELL'S EQNS ARE COMPLETELY SYMMETRIC:

$$\left.\begin{aligned} dF &= 0 \\ d{*}F &= 0 \end{aligned}\right\} \quad \begin{aligned} F &\leftrightarrow {*}F \quad \text{SYMMETRY} \\ \underline{E} &\leftrightarrow \underline{B} \end{aligned}$$

& it's not clear which eq comes from geometry & which comes from an action principle!

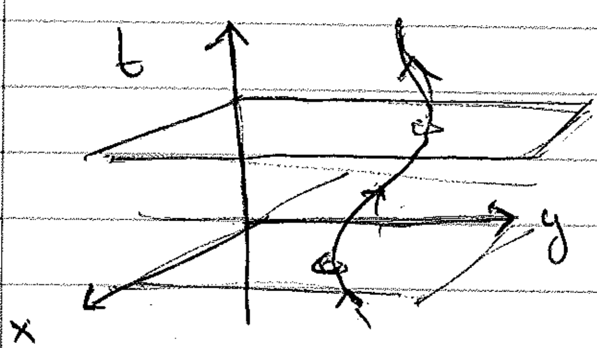
$\longrightarrow$ ELECTROMAGNETIC DUALITY

NOT TRUE IF WE HAVE $J \neq 0$ BUT TRY TO INCLUDE MAGNETIC CURRENT.

$$dF \overset{?}{=} J_m \cdot 4\pi \quad \longleftarrow \text{CAREFUL!!}$$
$$d{*}F = J \cdot 4\pi$$

$$dF = 4\pi J_m \Rightarrow d\,dA = 4\pi J_m \neq 0 \;?!$$

$\Rightarrow F \neq dA$ everywhere!



WORLDLINE OF A MAGNETIC MONOPOLE

$F = dA$ everywhere <u>BUT</u> the WORLDLINE

→ HAVE TO PUNCTURE A HOLE IN THE SPACETIME MANIFOLD THAT A LIVES ON.

THE MOMENT YOU PUNCTURE A HOLE, ALL "NICE" THINGS ARE <u>NOT</u> NICE!

↳ breaking Poincaré lemma

& now integrals of 4-forms (like $S = \int F \wedge {*F}$) pick up the boundaries at the punctures.

⇒ THE PHYSICS OF MAG. MONOPOLES CARES ABOUT THE <u>TOPOLOGY</u> OF THE SPACETIME MANIFOLD

CONSISTENCY OF THE RESULTING THEORY IMPOSES CONDITIONS ON J & J_M, among them is the famous dirac quant. cond:

$$\boxed{q_e q_m = \# \mathbb{Z}}$$

BIG q_e → SMALL q_m & VV.

EM DUALITY IS A PERT — NONPERT DUALITY.

GAUGE REDUNDANCY

GO BACK TO SIMPLE, monopole-free universe.

$F = dA$. $\quad$ A has a redundancy

GAUGE SYMMETRY

$\varphi \to \varphi + \dot{\alpha}$

$\underline{A} \to \underline{A} - \vec{\nabla}\alpha$

$$\boxed{A \to A + d\alpha}$$

because $dd\alpha = 0$, F is unchanged

this gauge redundancy gives freedom to pick nice gauges for EM problems.

BUT ALSO GIVES PROBLEMS: MORE MATHEMATICAL DEGREES OF FREEDOM THAN PHYSICAL.

eg. A PLANE WAVE $\sim \underbrace{A_\mu}_{\text{POLARIZATION}} e^{ik\cdot x}$

HAS 4 COMPONENTS...

ONLY 2 ARE PHYSICAL!

A^a_μ → 1 COMB is LEFT POLARIZED
1 COMB is RIGHT POLARIZED
1 —"— LONGITUDINAL, not present for massless photon
↳ EOM, for ex.
1 LEFTOVER
Gauge redundancy
→ the cost of COVARIANT DESCRIPTION.
(COVARIANCE: 4 COMPONENTS REQ. TO TRANSFORM WELL...)

THIS UNPHYSICAL DOF SHOWS UP IN SURPRISING PLACES...

eg. the STRONG CP PROBLEM

$$\mathcal{L} \supset \underbrace{G \wedge * G}_{\text{FIELD STRENGTH FOR STRONG FORCE}} + G \wedge G$$

SINCE G IS A 2-FORM, THIS HAPPENS TO BE A VALID LAGRANGIAN

in E&M, $F \wedge F$ turns out to be a total derivative ... integrating

$$\int_{\text{SPACETIME}} d(\cdots) = \int_{\text{BDRY}} (\cdots) = 0$$

EVERYTHING 0 @ ∞ OR ELSE ∞ ENERGY

BUT FOR STRONG FORCE, (non Abelian), TURNS OUT THAT $G \wedge G$ CAN BE NONZERO @ ∞ BECAUSE THE "UNPHYSICAL COMPONENTS" ARE NON ZERO.

→ contributes nothing to ENERGY, but gives an overall PHASE

$$Z = e^{iS} = \left(e^{iS_0}\right) e^{i\theta_{cp}}$$

FACT: PHYSICALLY, THIS PHASE WILL DIFFERENTIATE PARTICLES FROM ANTIPARTICLES ... SHOWS UP IN DIPOLE MOMENT OF NEUTRON.

→ no deviation found $\theta_{cp} < 10^{-11}$

→ HUGE OPEN QUESTION IN PHYSICS.