LEC 16: DIFFERENTIAL FORMS

28 OCT

→ "Vector calculus" is really "form calculus"

- SEE STONE & GOLDBART
- also: math.GT/0306194

ADV: A. MCINERNEY 1st STEPS IN DIFF GEO.
or: ANY GR TEXT.

REVIEW

BASIS OF DUAL VECTORS: dx^μ

s.t. $dx^\mu(\partial_\nu) = \delta^\mu{}_\nu$

BASIS OF VECTORS

$V = V^\mu \partial_\mu$

can act on a function $f: M \to \mathbb{R}$

to give directional derivative

HOW DO WE GET THESE dx^μ's?

DIFFERENTIAL OPERATOR / EXTERIOR DERIVATIVE

d: k-form → (k+1) form

so: a 0-form is just a function, f.
(has no form-ness.)

$$df = \frac{\partial f}{\partial x^\mu} \boxed{dx^\mu}$$

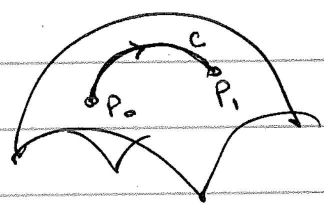
basis 1-form

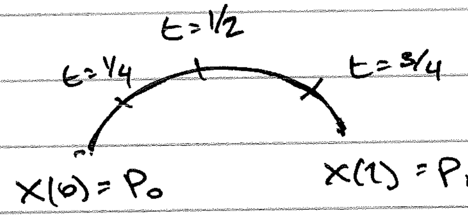
takes a vector V to give directional deriv.

df is a 1-form

BORN TO BE INTEGRATED: $\int_c dF = ?$

from ordinary calculus, you instinctively want to write: $\int_c df = f(P_1) - f(P_0)$





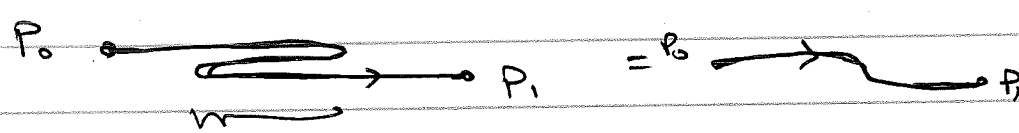
ONE WAY TO SEE THIS: PARAMETERIZE THE PATH: let $x : \mathbb{R} \to M$ s.t. $\begin{cases} x(1) = P_1 \\ x(0) = P_0 \end{cases}$

LIKE A TIME PARAM

then: $\int_c df = \int_0^1 \underbrace{\left(\frac{df(x(t))}{dt}\right)}_{} dt$

"signed integral"

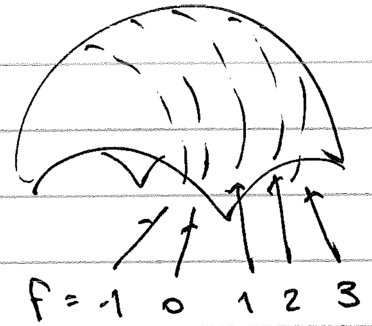
$$\frac{\partial f(x(t))}{\partial x^i} \underbrace{\frac{\partial x^i}{\partial t}}_{}$$



SWITCH-BACK CANCELS

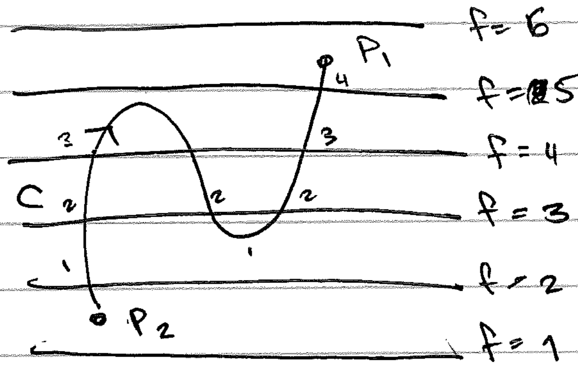
HW COMPARE THIS TO UNSIGNED INTEGRAL $\int \left|\frac{df}{ds}\right| ds$ FOR, eg, ARCLENGTH.

A MORE GEOMETRIC PICTURE: IMAGINE <u>CONTOURS</u> OF CONSTANT f ON M.



DASHED LINES: POINTS $x \in M$ s.t. $f(x) = -1, 0, 1, \ldots$

$\Downarrow$



$f(P_1) = 5.5$

$f(P_0) = 1.5$

GEOM. DEF

<u>then</u>: $\boxed{\int_C df = \#\text{ times we move up a contour} - \#\text{ times we move down}}$

$$\Rightarrow \int_C df = \underbrace{f(P_1) - f(P_0)}_{\text{ORIENTED BOUNDARY}} \quad \Big\} \text{ short cut.}$$

$$= f\big|_{\underset{\text{BOUNDARY}}{\partial C}}$$

nb McInerny is good

~~APPE~~ $ ~17.5

$$x^1 = x \qquad x^2 = y$$

2-forms

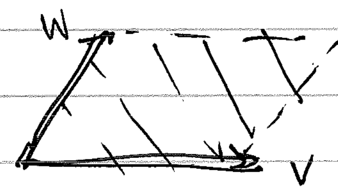
$$\omega = \frac{1}{2!}\,\omega_{\mu\nu}\, dx^\mu \wedge dx^\nu$$
$$= \tfrac{1}{2}\left[\omega_{xy}\, dx\wedge dy + \omega_{yx}\, dy \wedge dx\right]$$
$$= \tfrac{1}{2}\big[\underbrace{\omega_{xy} - \omega_{yx}}_{2\omega_{xy}\ \text{BY INDUCED ANTISYM.}}\big]\, dx\wedge dy \qquad \leftarrow dx\wedge dy = -dy\wedge dx$$
$$= \underbrace{\omega_{xy}\, dx\wedge dy}$$

MACHINE THAT TAKES 2 VECTORS & SPITS OUT ANTISYMMETRIC PRODUCT OF COMPONENTS

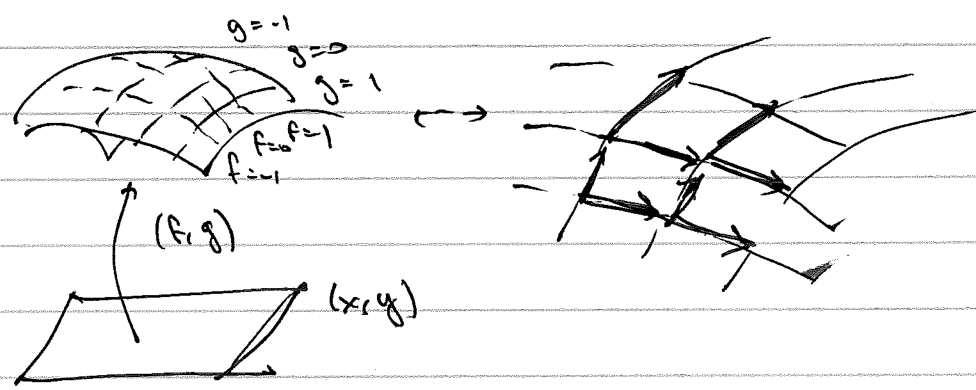
$$dx\wedge dy = dx\otimes dy - dy \otimes dx$$

$$dx\wedge dy(V,W) = V^1W^2 - V^2W^1$$

gives vol of parallelogram (ORIENTED)



if $\omega = d(f(x,y))\wedge d(g(x,y))$



2-form is infinitesimal area of a manifold. (ORIENTED)

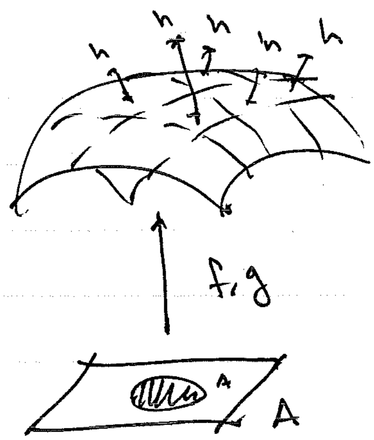
- 3-form is ∞'imal volume, also oriented

HW. 5b6
§ 12.2.2.

THIS STORY GENERALIZES

$$\int_A \underbrace{h(x,y)}_{\text{"HEAT MAP"}}\, df \wedge dg \qquad f, g : \mathbb{R}^2 \to M \qquad h : \mathbb{R}^2 \to \mathbb{R}$$

↑ weights EACH AREA ELEMENT

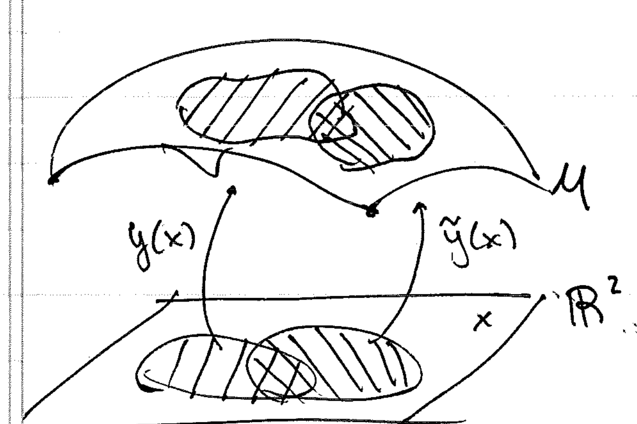


then: $\int_A h \, df \wedge dg$ is a 2D RIEMANN INTEGRAL OF h

induced metric p. 421

this is a nice way to PARAMETERIZE

ALSO GIVES A PICTURE FOR US TO EXPLORE A BETTER WORKING DEFINITION OF A MANIFOLD.



n-DIM SPACE WHICH IS EVERYWHERE LOCALLY (in patches) DIFFEOMORPHIC w/ $\mathbb{R}^n$ s.t. THESE MAPS AGREE WHERE THEY OVERLAP

eg. S^2, 2-SPHERE. ONE PATCH WON'T DO.

p. 420 s/o Go volumes

SPECIAL CASE: n-DIM MANIFOLD IS EMBEDDED IN A HIGHER-DIM EUCLIDEAN SPACE

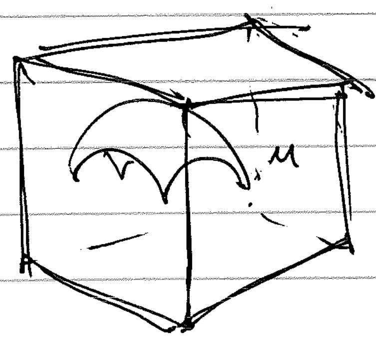
("normal" w/ usual sense of volumes ... ie w/ EUCL. METRIC

nb: (1) MANIFOLDS DON'T HAVE TO BE EMBEDDINGS

(2) YOU DON'T NEED METRIC TO INTEGRATE

↳ we didn't use it; not when we have differential forms

in this case:



not antisymm.

$$ds^2 = \sum_i^{m>n} (dx^i \otimes dx^i)$$

SURFACE M IS A RESTRICTION

so $x^i = x^i(y^1, \ldots, y^n)$

s/o Go p. 421

INDUCED METRIC :

$$ds^2 = \sum_i^{m>n} \left(\frac{\partial x^i}{\partial y^\mu} dy^\mu\right) \otimes \left(\frac{\partial x^i}{\partial y^\nu} dy^\nu\right)$$

$$= \underbrace{\left(\sum_i^{m>n} \frac{\partial x^i}{\partial y^\mu}\frac{\partial x^i}{\partial y^\nu}\right)}_{g_{\mu\nu} \text{ on } M} dy^\mu \otimes dy^\nu$$

HW:

IN THIS CASE, THE DIFFERENTIAL "VOLUME" is

$$d(\mathrm{Vol}) = \underbrace{\underbrace{\sqrt{\det g_{\mu\nu}}}_{\text{jacobian.}}\, dy^1 \wedge \cdots \wedge dy^n}_{\text{invariant under change of coordinates}}$$

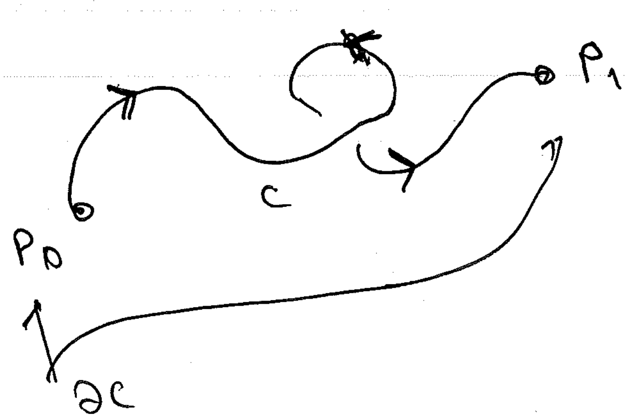
Stokes' theorem

INTEGRALS TAKE $d\omega$ over A "volume" to ω over a "surface area"

$$\int_V \underbrace{d\omega}_{(k+1)\text{ form}} = \int_{\partial V} \omega$$

(k+1)-dim MANIF.

k-DIM ENCL. AREA } ORIENTED



$$\int_C df = \int_{\partial C} f = f(P_1) - f(P_0)$$

2D CASE: CONSIDER $\omega = A_i\, dx^i = A_x dx + A_y dy + A_z dz$

$$d\omega = d(A_x dx) + \cdots$$

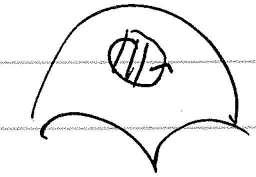
$$= \left(\frac{\partial A_x}{\partial x} \underbrace{dx \wedge dx}_{\substack{=0 \\ \text{by antisym}}} + \frac{\partial A_x}{\partial y} dy \wedge dx + \frac{\partial A_x}{\partial z} dz \wedge dx + \underbrace{A_x\, d\,dx}_{=0 \text{ by } d^2=0} \right)$$

$$= \left(-\frac{\partial A_x}{\partial y} dx \wedge dy + \frac{\partial A_x}{\partial z} dz \wedge dx \right) + \cdots$$

chk:
$$= \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) dx \wedge dy + \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) dy \wedge dz$$
$$+ \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) dz \wedge dx$$

RECOGNIZE: the components are $(\nabla \times A)_i$;
the 2-form basis in the last line are the (+ oriented) areas of parallelograms
$\Leftrightarrow$ normal vectors
($dx \wedge dy \sim \vec{dx} \times \vec{dy} = \vec{dz}$)

$$\int d\omega$$
$$\downarrow$$
$$\int_V d(A_i\, dx^i) = \boxed{\int (\vec{\nabla} \times \vec{A})\, dA}$$
$$\shortparallel$$
$$\int_{\partial V} \omega = \boxed{\oint \vec{A} \cdot d\ell}$$



Green's thm

SUGGESTIVE NAMING: $\vec{f}$

<u>3D CASE</u>: $\omega = f_x\, dy \wedge dz + f_y\, dz \wedge dx + f_z\, dx \wedge dy$

$$d\omega = \Big(\underbrace{\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial z} + \frac{\partial f_z}{\partial x}}_{\nabla \cdot \vec{f}}\Big)\, dx \wedge dy \wedge dz$$

$$\int_V d\omega = \int_V \nabla \cdot \vec{f}\ d(\mathrm{Vol})$$

$$\int_{\partial V} \omega = \int_{\partial V} \underbrace{\vec{f} \cdot d\vec{A}}$$

IDENTIFYING, eg $dy \wedge dz = \hat{n}_x\, dA$

FURTHER REMINDING US OF "VECTOR CALCULUS":

<u>FOR ω A 0-FORM</u> (function)

$d\omega$ is A 1-FORM $(\partial_x f\, dx + \cdots)$

$d^2\omega$ is A 2-FORM $[(\nabla \times (\nabla f))_x\, dy \wedge dz + \cdots]$

$= 0$ (eg b/c BNDY OF BNDY $= 0$)

$$\Rightarrow \quad \vec{\nabla} \times \vec{\nabla} = 0 \qquad [\text{curl} \circ \text{grad} = 0]$$

<u>FOR ω A 1-FORM</u> $\qquad \omega = f_x\, \cancel{dy \wedge dz}\ dx + \cdots$

$d\omega$ is A 2-FORM $(\vec{\nabla} \times \vec{f})_z\, dx \wedge dy + \cdots$

$d^2\omega$ is A 3-FORM $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{f})\ dx \wedge dy \wedge dz$

$= 0$

$$\Rightarrow \quad \vec{\nabla} \cdot \vec{\nabla} = 0 \qquad (\text{div} \circ \text{curl} = 0)$$

~~SOME~~ <u>CONTEXT</u>: we "generalized" vector calculus.

TO NON-FLAT SPACES

UNIFIED DIFFERENTIAL STRUCTURE

HIGHER DIM, ...

Killer app for vector calculus? E & M!

first, a quick result:

ω comes from POTENTIAL

<u>EXACT FORM</u>: $\omega = dA$

<u>CLOSED FORM</u>: $d\omega = 0$

OBVIOUS: EXACT $\Rightarrow$ CLOSED

not obvious: $\Leftarrow$

Poincaré lemma

→ holds for "nice" spaces (CONTRACTIBLE) } most of our cases.

POTENTIALS

Poincaré: SUPPOSE "VEC FIELD" $E_i dx^i$ s.t. $\nabla \times E = 0$

$$\rightarrow dE = 0 \Leftrightarrow E = d\varphi$$

$$\vec{E} = \vec{\nabla}\varphi$$

i.e. $\vec{E}$ HAS NO CURL $\Rightarrow$ $\vec{E}$ IS GRAD OF SCALAR.

HW: Maxwell's eq.

SIMILARLY: SUPPOSE "VEC FIELD" ~~[illegible]~~ $B_x\, dy \wedge dz + \cdots$

nb: very diff. object!

$$dB = 0 \Leftrightarrow B = dA$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{A}$$

on this note: an interesting operator

HODGE STAR: $*$ on n dim manifold

$$* dx^{i_1} \wedge \dots \wedge dx^{i_k} = \frac{1}{(n-k)!}\, \varepsilon_{i_1 \dots i_n} \underbrace{\cancel{[illegible]}}_{dx^{i_{k+1}} \wedge \dots \wedge dx^{n}}$$

turns 2-form $\longrightarrow$ 1-form in 3-SPACE.

SO: IN MINKOWSKI SPACE,
EM FIELDS LIVE IN $F_{\mu\nu}$

& one key point is $F = dA$

4- POTENTIAL
$A_\mu = (\varphi, \underline{A})$

next time: pushing vectors