TRANSFORMING
LEC 14: TENSORS

23 OCT

LAST TIME: TENSORS: $T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}$

MULTILINEAR MAPS

THE PUNCHLINE: TENSORS HAVE WELL-DEFINED TRANSFORMATION PROPERTIES

HOW DO THESE TRANSFORM?
SIMPLEST EXAMPLE: $\mathbb{R}^2$

$$\begin{pmatrix} v^1 \\ v^2 \end{pmatrix} \longmapsto R \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$$

ROTATION MATRIX, $R(\theta)$

$$(w_1 \; w_2) \longmapsto (w_1 \; w_2)\, R^T$$

$$R^T(\theta) = R(-\theta) = R^{-1}(\theta)$$

LOOKS LIKE: THINGS W/ UPPER & LOWER INDICES TRANSFORM OPPOSITELY

$$v^i \to R^i{}_j v^j \qquad \text{(IMPLICIT SUM OVER } j\text{!)}$$

$$w_j \to (R^{-1})_j{}^k w_k \qquad \text{(IMPLICIT SUM OVER } k\text{!)}$$

$$T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q} \to (R)^{i_1}{}_{i_1'} \cdots (R^{-1})_{j_1}{}^{j_1'} \cdots T^{i_1' \cdots i_p'}{}_{j_1' \cdots j_p'}$$

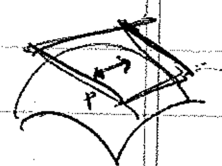
cf APPEL 16.5.6

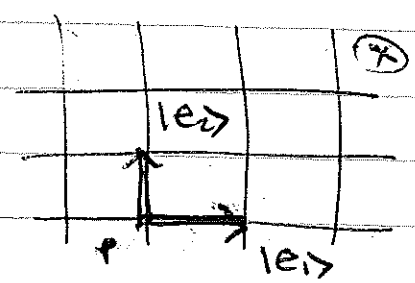
MORE GENERALLY, CONSIDER 2 DIFFERENT COORDINATE SYSTEMS; x & y, DESCRIBING THE SAME SPACE.

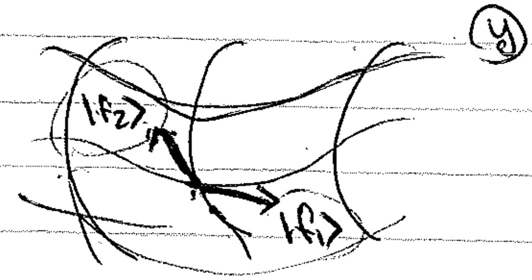
(analogous to fixing coordinates, but transforming space)

x, y COORD SYS $\longrightarrow$ $|e_i\rangle$ & $|f_i\rangle$

BASIS VECTORS OF TANGENT SPACE







CONSIDER AN INFINITESIMAL VECTOR AT P

$$|\varepsilon\rangle = \delta x^i |e_i\rangle = \delta y^j |f_j\rangle$$

HOW DOES δx RELATE TO δy?

IF YOU CAN WRITE $y = y(x)$ (CHANGE OF COORD)

then:

$$\boxed{\delta y^i = \frac{\partial y^i}{\partial x^j}\, \delta x^j}$$

HOW COORDS TRANSFORM

THE JACOBIAN FOR CHANGE OF COORDINATES

then : $\delta x^i |e_i\rangle = \frac{\partial y^j}{\partial x^j} \delta x^j |f_j\rangle$

CHANGE DUMMY INDICES $\longrightarrow$ $= \frac{\partial y^k}{\partial x^i} \delta x^i |f_k\rangle$

then $\rightarrow$ $\boxed{|e_i\rangle = \frac{\partial y^k}{\partial x^i} |f_k\rangle}$

HOW BASIS ELEMENTS TRANSFORM $\boxed{|f_i\rangle = \frac{\partial x^j}{\partial y^i} |e_j\rangle}$

y-COORD TANGENT SP.

x-COORD. TANGENT SPACE

INVERSE JACOBIAN

EVIDENTLY: TO GO FROM X-BASIS $\longrightarrow$ Y-BASIS

UPPER INDICES : $V^i \rightarrow \frac{\partial y^i}{\partial x^j} V^j$

LOWER INDICES: $W_i \rightarrow \frac{\partial x^j}{\partial y^i} W_j$

$= \left[\left(\frac{\partial y(x)}{\partial x}\right)^{-1}\right]_i{}^j$

n.b. this corroborates identification of
$|e_i\rangle = \partial/\partial x^i$

REMARKS

- the basis for the vectors is, evidently, a dual vector: $|e_i\rangle$ has a lower index
- the physical vector $|V\rangle = V^i |e_i\rangle$ has no indices.

when you transform

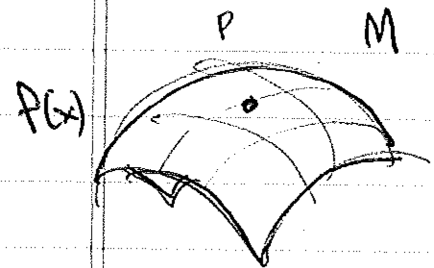
$$\left.\begin{aligned} V^i &\to (\partial y/\partial x)^i{}_j\, V^j \\ |e_i\rangle &\to (\partial x/\partial y)_i{}^j\, |e_j\rangle \end{aligned}\right\} |V\rangle \longrightarrow |V\rangle$$

$\Rightarrow$ PHYSICS DOESN'T CARE ABOUT YOUR COORDINATE SYSTEM

↳ nb if you do a physical transformation (rather than just new coordinates) then $|e_i\rangle$ fixed.

WHAT ABOUT DUAL VECTORS?
DO WE EVEN HAVE ANY GOOD EXAMPLES?

Here's a familiar one:



SUPPOSE YOU HAVE SOME FUNCTION f DEFINED OVER THE WHOLE SURFACE (MANIFOLD), M. $f: M \to \mathbb{R}$

CONSIDER THE DIFFERENTIAL OF f: df.

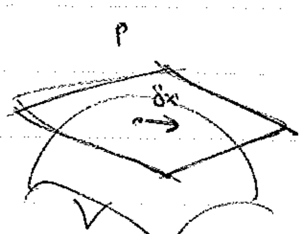
IN CARTESIAN COORDINATES,

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \cdots = \vec{\nabla} f \cdot d\vec{x}$$

THUS FAR, THIS IS STILL A WEIRD ABSTRACT OBJECT. I WANT IT TO MEAN: THE CHANGE IN SOME (PHYSICAL) QUANTITY f AS I MOVE FROM $p^i \to p^i + \delta x^i$

infinitesimal displacement is a tangent vec.

SO REALLY I WANT $df|_p = \frac{\partial f}{\partial x^i}\Big|_p dx^i + \cdots$ (or $d_p x$)



RECALL THAT $dx^i(\partial/\partial x^j) = \delta^i_j$

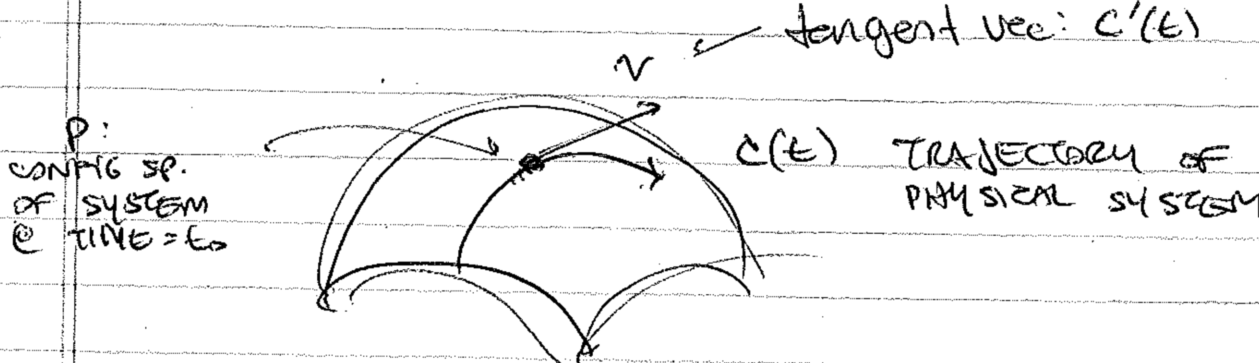
(MAP: $V \to \mathbb{R}$)

DUAL VEC IS EXACTLY WHAT WE WANT!

$d_p f$: object that tells us how f changes @ point p

v : tangent vector of M @ p
GIVES A DIRECTION.

$d_p f(v)$: DIRECTIONAL DERIVATIVE;
how f changes @ p
if it has a "velocity" of v



then $d_p f(v = c'(t))$
is how quickly f is changing.

$$d_p f = \frac{\partial f}{\partial x^i}\Big|_p \langle e^i_{(p)}| \quad dx^i @ p$$

HOW DOES IT TRANSFORM?

guess : upper index : $\partial y/\partial x$

lower index : $\partial x/\partial y$

$$\frac{\partial f}{\partial x^i} \longmapsto \frac{\partial x^j}{\partial y^i} \frac{\partial f}{\partial x^j} \quad \checkmark$$

UPPER INDEX IN DENOMINATOR = LOWER INDEX

& similarly :

$$\langle e^i| = dx^i \longrightarrow \frac{\partial y^i}{\partial x^j} dx^j$$

both of these just "make sense" from the chain rule.

do: w/ the simplest tensors

WE ARE HAPPY THAT UPPER INDICES & LOWER INDICES TRANSFORM OPPOSITELY.

GENERAL TENSOR TRANSFORMATION RULE

$$T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q} \to \left(\frac{\partial y}{\partial x}\right)^{i_1}{}_{i_1'} \left(\frac{\partial y}{\partial x}\right)^{i_2}{}_{i_2'} \cdots \left(\frac{\partial y}{\partial x}\right)^{i_p}{}_{i_p'}$$

$$\times \left(\frac{\partial x}{\partial y}\right)^{j_1'}{}_{j_1} \left(\frac{\partial x}{\partial y}\right)^{j_2'}{}_{j_2} \cdots \left(\frac{\partial x}{\partial y}\right)^{j_q'}{}_{j_q}$$

$$\times T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}$$

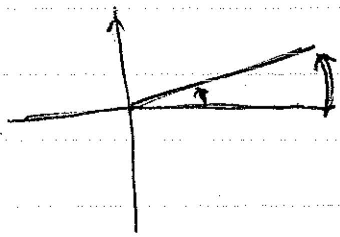
APPLICATION: GRADIENT IN ~~SPHERICAL~~ POLAR COORDINATES

from: physics.stackexchange

"derive the vector gradient in spherical coordinates from first principles"

NEED ORTHONORMAL BASIS

∂_θ, ~~∂_ϕ~~, ∂_r ARE ALWAYS ORTHOGONAL, BUT THE ANGULAR VECTORS HAVE LENGTH THAT DEPENDS ON POSITION.



same $\partial/\partial\theta$ gives different length displacement

∂_x, ∂_y ARE NORMALIZED

$$\frac{\partial}{\partial\theta} = \frac{\partial x}{\partial\theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\theta}\frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y}$$

$$x = r\cos\theta \qquad \frac{\partial x}{\partial\theta} = -r\sin\theta \qquad \frac{\partial x}{\partial r} = \cos\theta$$

$$y = r\sin\theta \qquad \frac{\partial y}{\partial\theta} = r\cos\theta \qquad \frac{\partial x}{\partial r} = \sin\theta$$

$$\frac{\partial}{\partial\theta} = -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y}$$

of course, the metric told us this!

$$\left|\frac{\partial}{\partial\theta}\right|^2 = r^2\sin^2\theta\left|\frac{\partial}{\partial x}\right|^2 + r^2\sin^2\left|\frac{\partial}{\partial y}\right|^2 = r^2$$

$$\underbrace{d_p f}_{\text{FORM}}(\underbrace{\delta r}_{\text{VEC.}}) = \frac{\partial f}{\partial x^i} \underbrace{dx^i(\delta r)}$$

$$= \sum_i \frac{1}{h_i} \frac{\partial f}{\partial x^i} \underbrace{h_i \delta r_i}$$

MULT & DIV. BY h_i

SUM. CONV. BREAKS DOWN

h_i = length of $\partial/\partial x^i$

$$\left\langle \frac{\partial}{\partial \theta} \middle| \delta r \right\rangle$$

then: $d_p f(\delta r) = \langle \nabla f | \delta r \rangle$

s.t. $\langle \nabla f | = \sum_i \frac{1}{h_i} \frac{\partial f}{\partial x^i} \left\langle \frac{\partial}{\partial x^i} \right|$

$$\boxed{\nabla f = \frac{\partial f}{\partial r} \left\langle \frac{\partial}{\partial r} \right| + \frac{1}{r} \frac{\partial f}{\partial \theta} \left\langle \frac{\partial}{\partial \theta} \right|}$$