LEC 13 : TENSORS

21 OCT

VECTOR SPACE, V : $V^* \to \mathbb{R}$

DUAL SPACE, V^* : $V \to \mathbb{R}$

vectors & dual vectors are LINEAR maps to $\mathbb{R}$ of one another

NEW NOTATION : height of index

$$|v\rangle = \sum_i \boxed{v^i}\, \underline{e}_{(i)} \qquad \text{or } |e_i\rangle \qquad \underline{e}_{(i)} = \partial/\partial x^i$$

JUST A NUMBER

$$\underline{e}_{(i)} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ i \\ \vdots \\ 0 \end{pmatrix}$$

CARRIES "MATRIX" STRUCTURE

$$\langle w| = \sum_j \boxed{w_j}\, \underline{e}^{*(j)} \qquad \underline{e}^{*(j)} = dx^j$$

$$\underline{e}^{*(j)} = (0\ 0 \cdots j \cdots 0) \quad \text{or } \langle e_j|$$

$$\langle w|v\rangle = \sum_i\sum_j v^i w_j \underbrace{\langle e^j|e_i\rangle}_{\delta^j{}_i} = \sum_i\sum_j v^i w_j \delta^j{}_i = \sum_i v^i w_i$$

NOTATION : REPEATED UPPER & LOWER INDEX $\Rightarrow$ SUM

$$\sum_i v^i w_i \longrightarrow v^i w_i$$

<u>MATRIX</u>: MULTILINEAR MAP : $V \times V^* \rightarrow \mathbb{R}$

alternately: $V \rightarrow V$

$$M^i{}_j \, \cancel{|e_i\rangle\langle e^j|} \quad v^k |e_k\rangle$$

$|e_i\rangle\langle e^j|$

(implicit $\sum_j \sum_i$) implicit $\sum_k$

$$= M^i{}_j v^k \, |e_i\rangle \underbrace{\langle e^j | e_k\rangle}_{\delta^j{}_k}$$

$$= \boxed{M^i{}_j v^j} \, |e_i\rangle$$

USUALLY WE JUST IGNORE THE BASIS BRAS & KETS

$\boxed{M^i{}_j v^j}$ is matrix multiplication

j is a dummy index

RESULTING OBJECT HAS ONE VECTOR INDEX

eg. $w_i \, M^i{}_j v^j = \#$

$(\cdots)\,(\vdots\vdots\vdots)\,(\vdots)$

eg: WHAT IS THE TRACE OF A MATRIX $M^i{}_j$?

$\rightarrow M^i{}_i$

<u>TENSOR</u>: GENERAL MULTILINEAR MAP

$$T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}$$

takes p dual vec, q vec $\to \mathbb{R}$

~~[illegible]~~ ~~CONTRACTIONS~~

<u>remark</u>: of course, $T^{i\cdots}{}_{j\cdots}$, is SHORTHAND FOR

$$T = T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q} \; |e_{i_1}\rangle \otimes \cdots \otimes |e_{i_p}\rangle \otimes \cdots \otimes \langle e^{j_1}| \otimes \cdots \otimes \langle e^{j_q}|$$

generaliz. of $|e_i\rangle\langle e^j|$

the $\otimes$ means direct product — they're just a reminder that these live in a different copy of the space

IE THESE BRAS & KETS NEVER HIT EACH OTHER eg. $T^{ij}{}_k S_l \longleftrightarrow$ combined: (2,2) tensor

$T^{ij}{}_k$: (2,1) tensor; S_l: (0,1) tensor

is this obvious?

YES: BUT THE NOT OBVIOUS THING IS WHAT TO CHECK! MULTILINEARITY.

eg. $T^{ij}{}_k S_{j}$

contract indices

end up w/ (1,1) tensor

REMARK: IN THIS WAY, CAN VIEW TENSORS AS MAPS BETWEEN PRODUCT SPACES

$$\text{eg}\quad T^{ij}{}_k : V \otimes V^* \otimes V^* \to \mathbb{R}$$
$$V \otimes V^* \to V$$
$$V \to V \otimes V^*$$

make sure you understand what $\otimes$ means:
eg if $V = \mathbb{R}^2$, then $V \otimes V \sim \mathbb{R}^4$ — really $\left(\begin{array}{c|c} & \circ \\ \hline \circ & \end{array}\right)\begin{pmatrix} (\\ (\end{pmatrix}$

<u>SPECIAL EXAMPLE</u>: T_{ij} any examples?

INNER PRODUCT $\longleftrightarrow$ "METRIC" (g_{ij})

$$g_{ij} : V \otimes V \to \mathbb{R}$$
$$V \to V^*$$

turns vec into dual

★ DEFINES AN OPERATION TO RAISE/LOWER INDICES

eg. $\underbrace{(g_{ij} V^j)}_{= V_i}$ s.t. $V_i W^i = g_{ij} V^i W^j$

<u>METRIC</u> (inner product):

$\underbrace{\text{LOWER INDICES}}$ OR $\underbrace{\text{TAKE 2 VEC} \to \#}_{\text{PREVIOUS DEF.}}$

↓

turn vec ⟷ dual

CAN ALSO DEFINE <u>INVERSE METRIC</u> g^{ij}

WHERE $\boxed{g^{ij} g_{jk} = \delta^i{}_k}$ $\left(= (\mathbb{1})^i{}_k \right)$

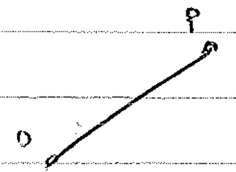
eg if $g_{ik} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$

$g^{ik} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$

eg if $g_{ik} = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

$g^{ik} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

METRIC - MEASURE of DISTANCE



infinitesimal arc length

$$ds^2 = dx^2 + dy^2 + dz^2$$

~~[illegible]~~: DUAL BASIS

e.g. $dx(\partial/\partial x) = 1$

$$ds^2 = dx \otimes dx + dy \otimes dy + dz \otimes dz$$

$\otimes$ ~~[illegible]~~ ...

$$= \langle e_x| \otimes \langle e_x| + \ldots$$

$$\equiv g_{ij}\, dx^i \otimes dx^j$$

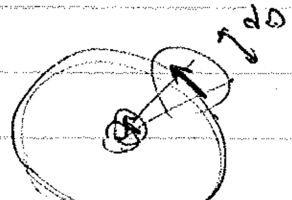
SPECIAL PROPERTY : SYMMETRIC ! $\_\_\_ \otimes \_\_\_$

DIFFERENT COORDINATES $\quad d\theta(\partial/\partial\theta) = 1$

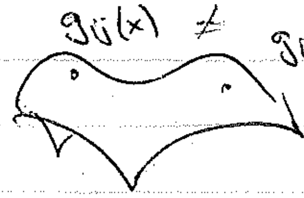
$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2$$

$$g_{ij} = \begin{pmatrix} 1 & & \\ & r^2 & \\ & & 1 \end{pmatrix} \qquad g^{ij} = \begin{pmatrix} 1 & & \\ & 1/r^2 & \\ & & 1 \end{pmatrix}$$

obs: tensors can be position-dependent!



generally: $g_{ij}(x) \neq g_{ij}(y)$



APPEL 455

So: tensors are generalized matrices

- UPPER/LOWER INDICES MATTER
  - V vs. V^* — LINEAR ($f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$)
- THEY ARE MAPS BETWEEN DIFFERENT PRODUCT SPACES

eg $T^{ij}{}_k$ takes 2 ~~vectors~~ dual v. & ~~dual~~ 1 vec to #

via $\underbrace{T^{ij}{}_k V_i W_j X^k}_{\#}$

eg $T^{ij}{}_k$ takes a (1 1)-tensor to a vector via

$\underbrace{T^{ij}{}_k S^k{}_j}_{\text{vector.}}$

- METRIC & ITS INVERSE IS A SPECIAL TENSOR: IT LETS YOU RAISE/LOWER INDICES & DEFINES THE MEASURE

$$ds^2 = g_{ij}\, dx^i \otimes dx^j$$

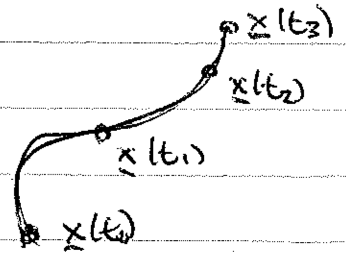
thus far: $ds = \sqrt{g_{ij}\, dx^i \otimes dx^j}$
gives a way to measure differential arc length... will have to generalize

- symmetry of indices can be important

& other line integrals

ASIDE: ARCLENGTH: usual method

DEFINE SOME "time" PARAMETER FOR TRAVERSING THE ARC



USUALLY PARAMETERIZE s.t. $t \in [0,1]$

WANT: INTEGRATE OVER THIS "TIME"

$$\int_0^1 \frac{ds}{dt}\, dt = \int_0^1 \sqrt{g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}}\; dt$$

note: $g_{ij} = g_{ij}(x)$!

WHAT'S SO GREAT ABOUT TENSORS?

they have well defined transformation properties.

change of coordinates or actual action on physical sys.

REMINDER: transf. of (row & col.) vectors in $\mathbb{R}^2$

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \longrightarrow \underbrace{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}}_{R} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

What about "dual vector"?

$$(w_1\ w_2) \longrightarrow (w_1\ w_2) \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{R^T}$$

INTUITION: 2 WAYS

1. USE METRIC TO TURN ROW VECTOR $\to$ COL VEC.

$$(w_1\ w_2) \overset{T}{\Rightarrow} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}^T$$

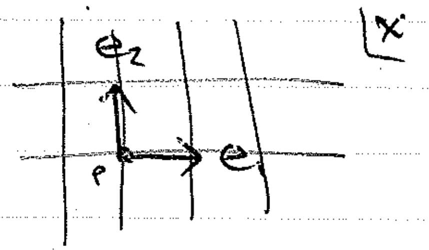
$$\hookrightarrow R \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}^T \overset{T}{\Rightarrow} (w_1\ w_2) R^T$$

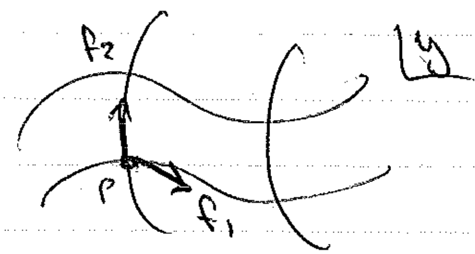
2. USE INNER PROD: for ANY $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, $(w_1\ w_2)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ IS INVARIANT. SINCE $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \to R\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$,

$$(w_1\ w_2) \to (w_1\ w_2)\, R^T$$

s.t. $(w_1\ w_2)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \to (w_1\ w_2) \underbrace{R^T R}_{\mathbb{1}} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

MORE GENERALLY: IMAGINE 2 COORD SYSTEMS





consider infinitesimal vector @ p

$$|e\rangle = \delta x^i |e_i\rangle = \delta y^j |f_j\rangle$$

to leading order:

$$\delta y^j = \underbrace{\boxed{\frac{\partial y^j}{\partial x^i}}}_{\text{Jacobian}} \delta x^i$$

OR: $\cancel{\delta x^i} |e_i\rangle = \frac{\partial y^j}{\partial x^i} \cancel{\delta x^i} |f_j\rangle$

$$\Rightarrow \quad |f_j\rangle = \boxed{\frac{\partial x^i}{\partial y^j}} |e_i\rangle$$

this justifies the identification of TANGENT BASIS VECTORS w/ DIFF. OPS

$$\underbrace{\boxed{\frac{\partial}{\partial y^j}}}_{|f_j\rangle} = \left(\frac{\partial x^i}{\partial y^j}\right) \underbrace{\boxed{\frac{\partial}{\partial x^i}}}_{|e_i\rangle}$$