LEC 12: REVIEW & intro to GEOMETRY 19 OCT.

OVER THE LAST 4 WEEKS – the important stuff

NEXT 5 WEEKS - special topics

1. GEOMETRY (calculus)
2. GROUP THEORY

THUS FAR

VECTOR SPACES & DUAL SPACES

QM

LINEAR ALGEBRA

↓

FUNCTION SPACES ← inner product (L^2 norm)

& DIFFERENTIAL OPERATORS AS MATRICES ON FUNCTION SPACE — self adjointness

DIFFERENTIAL EQUATIONS

↓

Solution by Green's Function

↕

propagator of physical effects

finding Green's Functions by Fourier transform

→ COMPLEX ANALYSIS AS A TOOL

↳ ANALYTIC/MEROMORPHIC FUNCTIONS ← nonanalyticity as N.P.

→ RESIDUE THEOREM

→ CONTOUR INTEGRAL GYMNASTICS

APPLICATION: CAUSALITY ↔ Green's f'n, KRAMER-KRONIG

OVERVIEW: DIFFERENTIAL GEOMETRY

calculus

METRIC ↔ INNER PRODUCT

"measure"

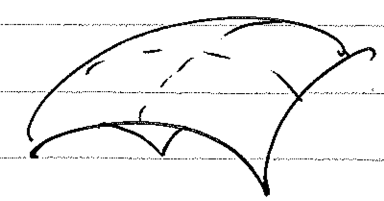
eg $d^n x = r^{n-1} dr \, d\Omega_{n-1}$

eg $d(\cos\theta)\, d\varphi$

weird "not flat" coordinates

Generalization of calculus to "weird, not-flat" spaces

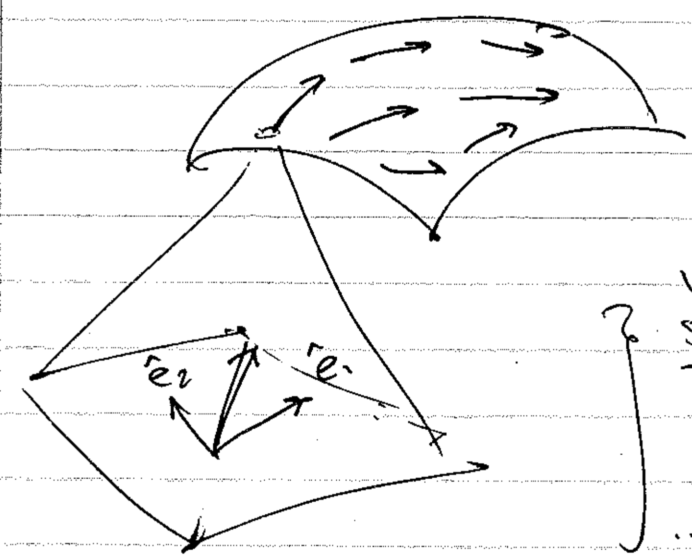
not flat ↔ metric



HOW TO DESCRIBE THIS.

~~[illegible]~~

VECTORS ON THESE SPACES



ACTUALLY, VECTOR **FIELDS**

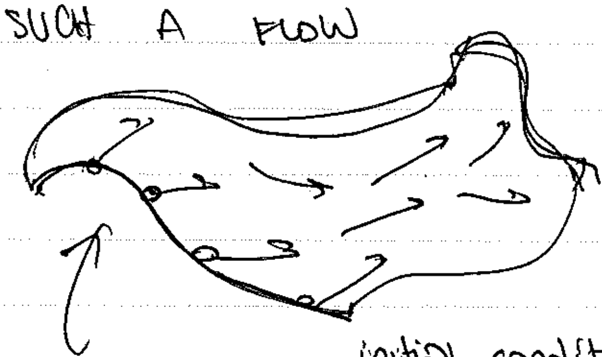
VECTORS @ A POINT LIVE IN SPACE THAT IS TANGENT TO SURFACE

... ∃ SUCH A TANGENT SPACE @ EACH POINT.

⇝ BUNDLE.

VECTOR FIELDS DEFINE A <u>FLOW</u> ON THE MANIFOLD.

↳ in fact: DYNAMICS $\left(\frac{d}{dt}\right)$ ARE SUCH A FLOW

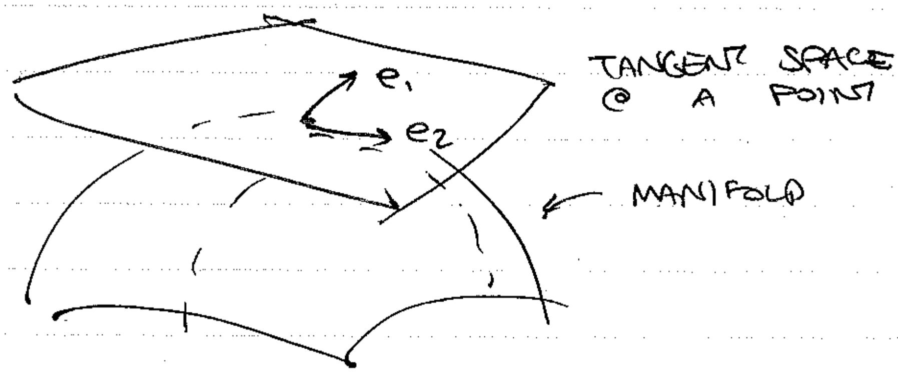


close relation of HAMILTONIAN MECHANICS TO GEOMETRY

this is like a differential eqn. & we "integrate it"

→ PHYSICS IS ABOUT THIS DYNAMICS

in fact: there is an <u>ISOMORPHISM</u> between DIFFERENTIAL OPERATORS & VECTOR FIELDS



$e_1 \sim \partial/\partial x$

$e_2 \sim \partial/\partial y$

} PARTIAL DERIVATIVES ↕ BASIS OF TANGENT SPACE if ∂_x is a vector, what is <u>DUAL</u> ?

$\rightsquigarrow$ SOMETHING RELATED TO INTEGRATION

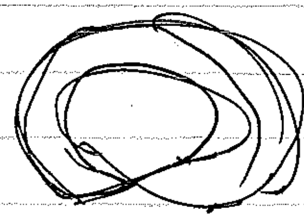
ANSW: <u>DIFFERENTIAL FORM</u>

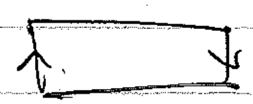
recall: DUAL: LINEAR MAP FROM VECTOR SPACE $\rightarrow \mathbb{R}$ ($\mathbb{C}$)

DIFF. FORMS ARE OBJECTS THAT ARE BORN TO BE <u>INTEGRATED</u>.

integration $\longleftrightarrow$ sense of global properties of manifold.

LINK TO TOPOLOGY

eg  Mobius strip

 vs. 

(TORUS)

<u>FORMS</u>: ALSO GIVE A SENSE OF (CO)-HOMOLOGY

$$\int d\omega = \omega|_{\underline{BNDY}}$$

in fact: more than just linear algebra
→ forms & vectors on manifolds
introduce the idea of
MULTILINEAR ALGEBRA
↕
TENSORS
↳ objects that are defined by how they transform under symmetries
↑
GENERALIZ. OF MATRICES

Symmetry is another very big idea in physics & it is important to be able to describe them mathematically
↓
GROUP THEORY → FINITE
→ CONTINUOUS
ALSO A MANIFOLD!

AN INCOMPLETE SURVEY of WHO CARES

METRIC/MANIFOLD/MEASURE ← change of coords

(SPECIAL) RELATIVITY

TENSORS → eg. EM FIELD STRENGTH
symmetry properties: rotations & boosts
→ WHY RELATIVITY ⇒ MAXWELL

FORMS: GAUGE THEORY → what is E&M?
(REDUNDANCY IN OUR THEORY
→ MAGNETIC MONOPOLES
(& their exotic cousins)

falling cat

THERMODY. POTENTIALS

→ TOPOLOGY

FLOW: HAMILTONIAN MECHANICS } COTANGENT BUNDLE
(CANONICAL FORMALISM)

Holonomic/Non-Holonomic sys.

~~[illegible]~~ not in this class: Riemannian Geometry
→ PROB

SYMMETRY: WHAT IS A PARTICLE?
SPIN & IT'S COUSINS, SPINORS

TENSORS: MULTILINEAR MAPS

goal: establish some notation, stay grounded in linear algebra before going to calculus.

VECTORS, once again (FINITE DIM, easy case)

SUPPOSE WE HAVE SOME VECTOR SPACE V OF DIMENSION n

$\hookrightarrow$ $v \in V$ CAN BE WRITTEN $\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$

~~NOTATION~~

WE ALSO HAVE A DUAL VECTOR SPACE V^*. THESE ARE LINEAR MAPS $V^*: V \to \mathbb{R}$.

eg BRAS. $\langle w| \in V^*$ turns $|v\rangle \in V$ into a number via $\langle w|v\rangle$.

now we're going to go back to row & column vector notation

NOTATION (Jay's favorite)

$|v\rangle \longrightarrow v^i$ ← i might call these v^μ

$\langle w| \longrightarrow w_j$ & w_μ ...

$\langle w|v\rangle = \sum_i w_i v^i \longmapsto w_i v^i$

any REPEATED PAIR of upper & lower indices are summed over

eg. $w^i x^j z_k y_j = w^i z_k \underbrace{\sum_j y_j x^j}$ $= w^i z_k \langle y|x\rangle$

each element is just a number so order doesn't matter

eg $w^i x^i$ ⇒ MEANS NOTHING
(or means nothing usefull

eg. $w^i x_i y^i$ ⇒ MEANS NOTHING

eg. MATRIX MULTIPLICATION
↳ MATRIX: $V \to V$, LINEAR
BASIS WERE THINGS LIKE $|e_i\rangle\langle e_j|$
b/c $(|e_i\rangle\underbrace{\langle e_j|)\,|\psi\rangle}_{\text{GIVES \#}}$
LEAVES VECTOR

SO A MATRIX $\in V \otimes V^*$
tensor product

means:
$$M^i{}_j$$

thus MATRIX MULTIPLICATION in SUMMATION NOTATION is

$$M^i{}_j V^j = \sum_j M^i{}_j V^j$$

$$= \begin{pmatrix} \circ & \circ & \circ \\ & & \\ & & \end{pmatrix} \begin{pmatrix} \circ \\ \circ \\ \circ \end{pmatrix} = \begin{pmatrix} M^1{}_1 V^1 + M^1{}_2 V^2 + \cdots \\ \\ \end{pmatrix}$$

EASY TO GENERALIZE

$$M_{ij} : V \to V^*$$

takes column vector to row vector

(vector)	(dual vector)
(covariant)	(contravariant)
(ket)	(bra)

any example of thing like this?

the <u>INNER PRODUCT</u>!

$\langle \,|\, \rangle$ rule to turn kets $\to$ bras s.t. you can dot them.

bra ket

eg. $f, g \in$ HILBERT SPACE

$$\langle f|g\rangle = \int \underbrace{f^*}_{} g \, dx$$

rule to turn $|f\rangle \to \langle f|$

SO INNER PRODUCT $\leftrightarrow$ g_{ij} (<u>METRIC</u>)

MORE GENERALIZATION

or (p,k)?

$T^{i_1 \cdots i_k}{}_{j_1 \cdots j_p}$ is a (k,p) tensor

s.t. $T : \underbrace{(V^* \times \cdots \times V^*)}_{k} \times \underbrace{(V \times \cdots \times V)}_{p} \to \mathbb{R}$

this is a multilinear map

$T^{ij}{}_k$ takes 2 dual vectors, 1 vector

(2,1)-tensor into a number

DUAL VEC

VEC.

$T^{ij}{}_k \, V_i W_j X^k$

$T^{ij}{}_k (V_i + Z_i) W_j X^k = T^{ij}{}_k V_i W_j X^k + T^{ij}{}_k Z_i W_j X^k$

& similar rules w/ scaling, etc.

What's so great about tensors?

WELL DEFINED TRANSFORMATION PROPERTIES

$V^i \to R^i{}_j V^j$ for ROTATION

$T^{ij}{}_k \to R^i{}_l R^j{}_m \underline{\underline{(R^T)^n{}_k}} T^{lm}{}_n$