LEC 11: DISPERSION RELATIONS — 17 OCTOBER

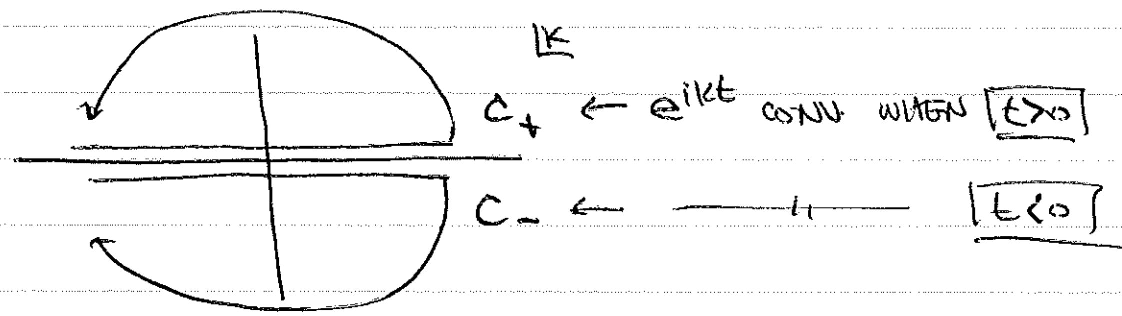
AGENDA:

- EULER Γ function → DISCUSSION OR IF WE HAVE TIME
- CORRECTION – MY FOURIER SIGNS
- DISPERSION

CORRECTION TO LAST TIME

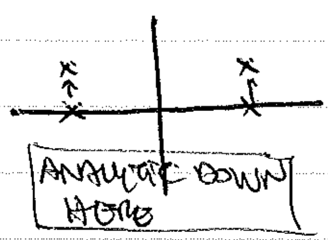
FOURIER CONVENTION: ← note: CHOICE OF SIGN IS CONVENTION

$$G(t) = \int \tilde{G}(k)\, e^{ikt}\, dk$$



GOOD GREEN'S FUNCTION: $G^{(r)}$ ⟷ CAUSAL: $G^{(r)}(t<0) = 0$, $G^{(r)}(t>0) \neq 0$

SO WE PUSH THE POLES INTO UPPER HALF PLANE s.t. the $t>0$ CONTOUR PICKS UP RESIDUES WHILE $t<0$ GETS NOTHING.



⤳ this is the main takeaway from last week!

n.b. if we used $G(t) = \int \tilde{G}(k)\, e^{-ikt}\, dk$,
then we swap the $\cancel{G^{(a)} \leftrightarrow G^{(r)}}$ $C_+ \longleftrightarrow C_-$ contours
($\&$ the $G^{(a)} \longleftrightarrow G^{(r)}$ pole conventions)

so: in what follows (& apparently what's the standard convention), WE DEFINE THE FOURIER TRANSFORM w/rt TIME AS

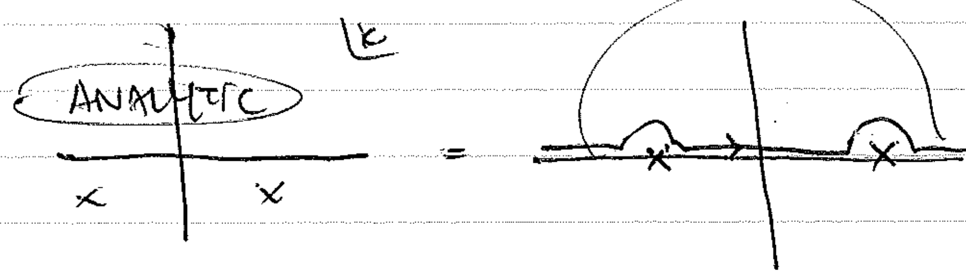
$$f(t) = \int \tilde{f}(k)\, e^{-ikt}\, dt$$

[note: THIS IMPLIES A + SIGN FOR THE SPATIAL TRANSFORM SINCE: $f(x) = \int \tilde{f}(k)\, e^{-ik^\mu x_\mu}\, d^4x$]

<u>IN THIS CONVENTION</u>:

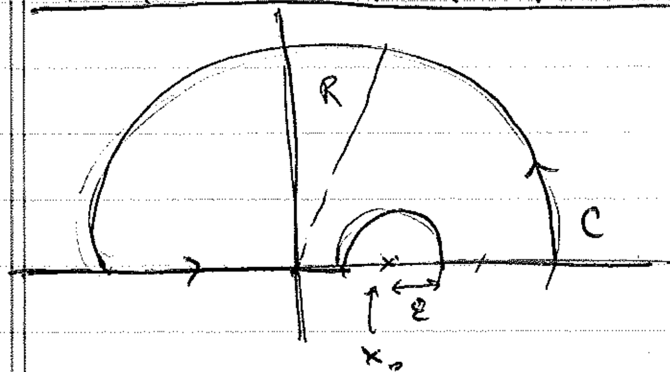
CAUSAL $\longleftrightarrow$ POLES SHIFTED <u>DOWN</u>

$\longleftrightarrow$ ANALYTIC IN UPPER HALF PLANE



APPROACH THIS FROM PRINCIPAL VALUE P.O.V.

nb. if f were analytic everywhere, it would be a boring function



SUPPOSE $f(z)$ ANALYTIC IN UPPER HALF PLANE (like a good causal Green's function!)

$\uparrow$ $f(z) \to$ on UPPER ARC; ie $f(Re^{iz}) \xrightarrow[R\to\infty]{} 0$

CAUCHY THM: $\oint_C \frac{f(z)\,dz}{z-x_0} = 0$

$\uparrow$ b/c $\frac{1}{z-x_0}$ IS ANALYTIC OVER THIS REGION.

$$\oint_C \frac{f(z)\,dz}{z-x_0} = \Big(\underbrace{\int_{-\infty}^{x_0-\varepsilon} + \int_{x_0+\varepsilon}^{\infty}}_{\text{PRINCIPAL VALUE}} + \int_{\text{SMALL ARC}} + \underbrace{\int_{\text{BIG ARC}}}_{=0 \text{ BY ASSUMP}}\Big) \frac{f(z)\,dz}{z-x_0}$$

$z = \varepsilon e^{i\theta} + x_0$

$$\int_\pi^0 \frac{f(\varepsilon e^{i\theta}+x_0)}{\cancel{\varepsilon e^{i\theta}}} \cancel{\varepsilon} i \cancel{e^{i\theta}}\, d\theta$$

$= -i\pi f(x_0)$

~~nb. if~~

$$\boxed{P\int_{-\infty}^{\infty} \frac{f(x)}{x'-x_0}\,dx' = i\pi f(x_0)}$$

DECOMPOSE INTO $\mathbb{R}$ & $\mathbb{Im}$ PARTS : $f = u + iv$

$$\mathcal{P}\int \frac{u(x) + i v(x)}{x - x_0}\, dx = i\pi u(x) - \pi v(x)$$

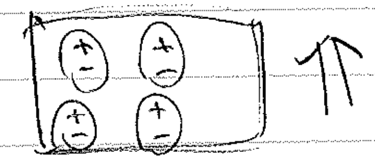
$$\Rightarrow \begin{cases} u(x) = \frac{1}{\pi}\,\mathcal{P}\int \frac{v(x)}{x - x_0}\, dx \\ v(x) = \frac{-1}{\pi}\,\mathcal{P}\int \frac{u(x)}{x - x_0}\, dx \end{cases}$$

↳ w/ kind of like an integrated version of the Cauchy-Riemann eqs.

the $\mathbb{R}$ & $\mathbb{Im}$ parts are related!

↳ also gets back to the notion that analytic functions "want" to be single variable, but $\mathbb{C}$ plane is 2D.

RECALL: EM WAVES IN DIELECTRICS



$$\underline{\nabla} \cdot \underline{E} = 4\pi(\rho_{FREE} + \rho_{BOUND})$$

WANT TO TREAT THIS AS "ENVIRONMENT"

$$= -\underline{\nabla} \cdot \underline{P} \quad \text{(POLARIZATION)}$$

$$\Rightarrow \underline{\nabla} \cdot (\underbrace{\underline{E} + 4\pi \underline{P}}_{\underline{D}}) = 4\pi \rho_{FREE}$$

$\underline{D}$ (DIELECTRIC DISPLACEMENT)

for ~~[illegible]~~ not-too-big $\underline{E}$, many materials obey

$$\underline{P} = \chi \underline{E}$$

χ ← ELECTRIC SUSCEPTIBILITY

$$\Rightarrow \underline{D} = \underbrace{(1 + 4\pi\chi)}_{\varepsilon}\underline{E}$$

ε ← DIELECTRIC CONST.

FOR AN EM WAVE PASSING THROUGH THE MEDIUM THE VALUE OF χ (OR ε) IS FREQUENCY DEPENDENT.

↳ PHYSICALLY: $\exists$ time scale for the molecules to flip

$$\boxed{P(\omega) = \chi(\omega) E(\omega)}$$ ← UP TO FACTORS OF 2π

IN FACT, χ IS A GREEN'S FUNCTION:

$$\underline{P}(t) = \int \chi(t-t')\, \underline{E}(t')\, dt'$$

CHK

$$\int P(\omega) e^{i\omega t}$$

$$= \int dt' \int d\omega\, e^{i\omega(t-t')} \chi(\omega) \int d\omega' e^{i\omega' t'} E(\omega')$$

$$= \int dt'\, d\omega\, d\omega'\, e^{i(\omega'-\omega)t'} e^{i\omega t} \chi(\omega) E(\omega')$$

$$\int dt'\, e^{i(\omega'-\omega)t'} = 2\pi\, \delta(\omega'-\omega)$$

$$= \int d\omega\, e^{i\omega t} \chi(\omega) E(\omega) \;//$$

WHAT DO WE KNOW ABOUT χ (AS A GREEN'S F'N)?

- $\chi(t<0) = 0 \longleftrightarrow$ ANALYTIC IN UHP

$$\Rightarrow \boxed{\begin{aligned} \mathrm{Re}(\chi(\omega)) &= \frac{1}{\pi} P\int \frac{\mathrm{Im}[\chi(\omega')]}{\omega'-\omega}\, d\omega' \\ \mathrm{Im}(\chi(\omega)) &= \frac{-1}{\pi} P\int \frac{\mathrm{Re}[\chi(\omega')]}{\omega'-\omega}\, d\omega' \end{aligned}}$$

SO WHAT

HECHT & ZAJAC. e.g. p. 23

EM WAVE $\sim e^{ikx - i\omega t}$

$k = \omega / v \leftarrow$ velocity (units of c)

BUT: $n = \frac{1}{v} = \sqrt{\varepsilon \mu} \rightarrow \sqrt{\varepsilon} \quad (\mu = 1)$ "PURE" DIELEC.

NOW $\varepsilon = 1 + 4\pi\chi$ HAS Re & Im PART

FREQ. DEP: DISPERSION
DIFF. WAVELENGTHS
SEPARATE.

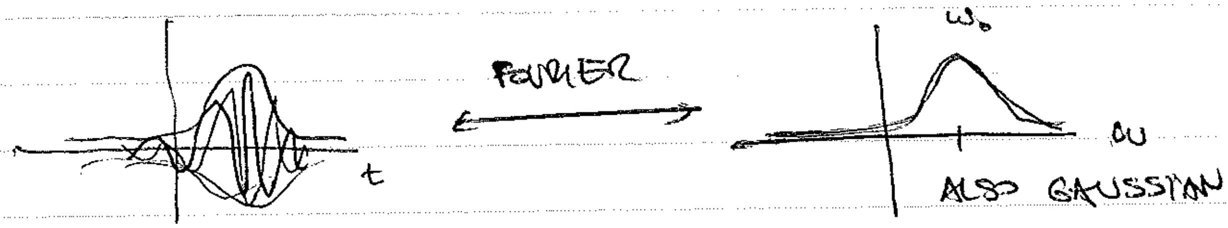
$e^{i(ik)x} \rightarrow e^{-kx}$
DISSIPATION
ENERGY LOST TO
THE MEDIUM.

So KRAMERS-KRÖNIG:
CAUSALITY/ANALYTICITY $\Rightarrow$ RELATES DISPERSIVE & DISSIPATIVE PARTS OF PROPAGATION

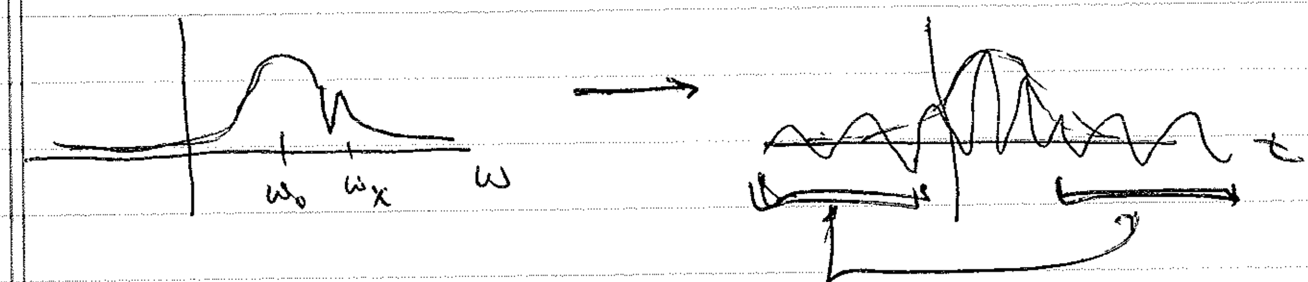
~~[illegible]~~?
APPLICATION: MEASURING INDEX OF REFRACTION BY MEASURING ABSORPTION.

BUT <u>WHY</u> DO WE EXPECT THIS?

IMAGINE GAUSSIAN WAVEPACKET



NOW SEND THROUGH IDEALIZED ABSORBING MATERIAL THAT EFFECTIVELY PULLS OUT ONE FREQUENCY, ω_x



NO LONGER A NICE WAVE PACKET!
GET THESE A-CAUSAL WAVES!

KRAMERS-KRÖNIG IS TELLING US THAT DISSIPATION (ABSORPTION) DOESN'T HAPPEN BY ITSELF → MUST COME W/ DISPERSION s.t. YOU DO NOT GET THIS ~~A-CAU~~ NON-CAUSAL JUNK!

TO FURTHER UNDERSTAND, IT'S USEFUL TO SEPARATE χ INTO EVEN & ODD PIECES W/RT TIME REVERSAL:

$$\begin{aligned} \chi_E &= \tfrac{1}{2}\left(\chi(t) + \chi(-t)\right) \\ \chi_O &= \tfrac{1}{2}\left(\chi(t) \underbrace{- \chi(-t)}\right) \end{aligned} \Bigg\} \text{ in time domain}$$

for $t>0$, these parts cancel in a causal function!

freq. space

$$\chi(\omega) = \int e^{i\omega t}\,\chi(t)\,dt$$

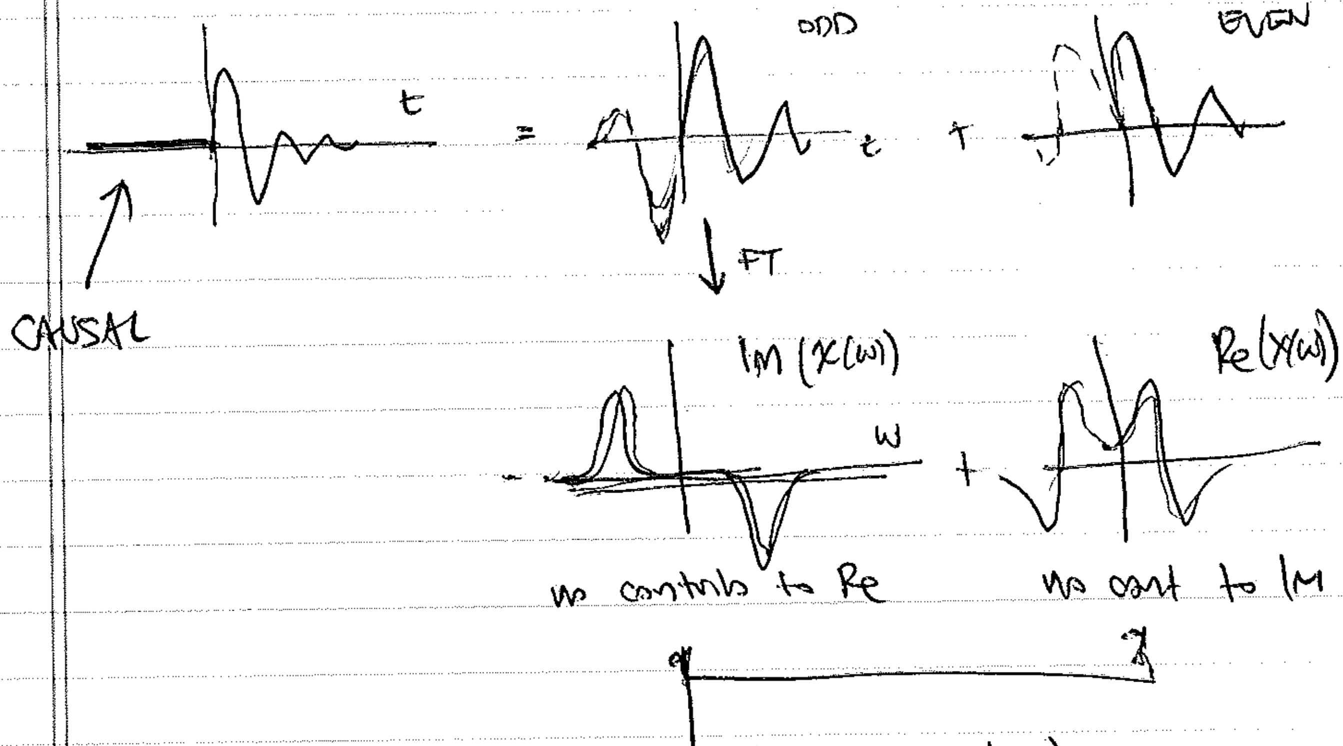
$$\underbrace{\cos\omega t}_{\text{EVEN IN } t} + i\,\underbrace{\sin\omega t}_{\text{ODD IN } t}$$

BUT $\int (\text{EVEN})(\text{ODD}) = 0$

$$\Rightarrow \begin{aligned} \operatorname{Re}\chi(\omega) &= \int e^{i\omega t}\,\chi_E(t)\,dt \\ \operatorname{Im}\chi(\omega) &= \int e^{i\omega t}\,\chi_O(t)\,dt \end{aligned}$$

THIS IS WHAT KRAMERS-KRONIG IS DOING

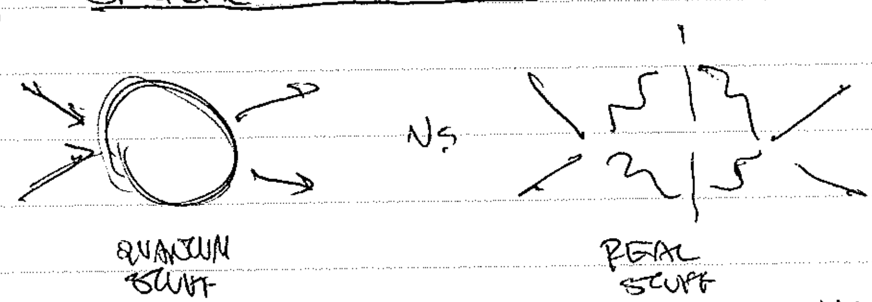
Sketch from WIKIPEDIA



Re & Im linked
to one another.

TIES INTO LOTS OF OTHER THINGS

eg OPTICAL THEOREM



→ CAN TAKE YOU
OUT OF YOUR
HILBERT SPACE

→ ANALOGOUS TO DISSIPATION

$\mathrm{Im}\, M \sim \sigma_{FWD}$