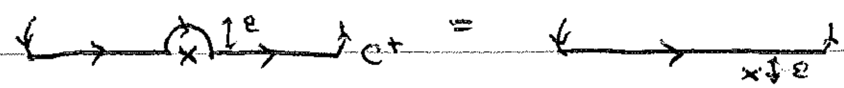
SOURCE: APPEL 8.1c, 15.2

LEC 10: GREEN'S FUNC & ℂ METHODS — 14 OCT
REVIEW; CAUCHY PRINC', Γ fun.

REMINDER: CAUCHY PRINCIPAL VALUE

What do we do when our integration hits a pole? → HAVE TO GO AROUND

eg 

POLE ON ℝ, DEFORM CONTOUR ABOVE/BELOW — DEFORM POLE BELOW / ABOVE

eg for $f(z) = \dfrac{F(z)}{z-z_0}$ ← ANALYTIC & $F(|z|\to\infty) \sim \frac{1}{|z|^{\#}}$; ← SIMPLE POLE @ $z_0 = x_0$

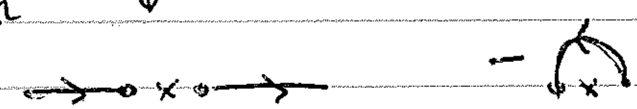
$$\int_{C+} f(z)\,dz = \int_{-\infty}^{\infty} \frac{F(x)}{x - x_0 + i\varepsilon} + \underbrace{\text{ARC}}_{\to 0 \text{ BY ASSUMP}}$$

↑ CONTOUR GOING ABOVE

$$= \left(P\int_{-\infty}^{\infty} \frac{F(x)}{x-x_0}\right) - i\pi \underbrace{F(x_0)}_{\text{RES.}}$$

VS. 2π

PRINCIPAL VALUE

CUT OUT POLE → 

REMINDER: $z = x_0 + \varepsilon e^{i\theta}$

$$\int_{\pi}^{0} F(x_0+\varepsilon e^{i\theta})/\varepsilon e^{i\theta}\; d(x_0+\varepsilon e^{i\theta})$$
$$= -\int_0^{\pi} F(x_0)\, i\,d\theta$$
$$= -i\pi F(x_0)$$

SIMILARLY: $\displaystyle\int_{C-} f(z) = P\int_{-\infty}^{\infty} \frac{F(x)}{x-x_0} + i\pi F(x)$

↑ CONTOUR GOING UNDER.

CAN THINK ABOUT THIS AS:

$$\left\langle \frac{1}{x-x_0+i\varepsilon}, F \right\rangle = \left\langle \mathcal{P}\frac{1}{x-x_0}, F \right\rangle - i\pi \left\langle \delta(x-x_0), F \right\rangle$$

$$\hookrightarrow \text{"} \frac{1}{x-x_0 \pm i\varepsilon} = \mathcal{P}\left(\frac{1}{x-x_0}\right) \mp i\pi\,\delta(x-x_0) \text{"}$$

symbolically! (AS A DISTRIBUTION)

EXPLICIT PROBLEM: GREEN'S FUNCTION OF HARMONIC OSC

nb APPEL USES FUNNY FOURIER NOTATION... BUT IT'S ACTUALLY VERY SENSIBLE

$\mathcal{O}: (d/dt)^2 + \omega_0^2$ — set $t_0 = 0$

GREEN'S FUNC. EQ: $G''(t) + \omega_0^2 G(t) = \delta(t)$

Strategy: FOURIER

WRITE $G(t) = \int_{-\infty}^{\infty} e^{ikt}\,\tilde{G}(k)\,dk$ (FOURIER TRANSFORM)

then: $\left[\left(\frac{d}{dt}\right)^2 + \omega_0^2\right] G(t) = \int_{-\infty}^{\infty} e^{ikt}\,(\omega_0^2 - k^2)\,\tilde{G}(k)\,dk$

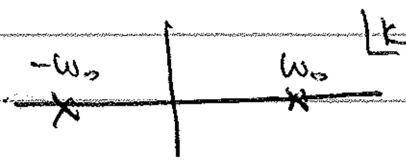
RHS: $\int_{-\infty}^{\infty} e^{ikt}\,dk \Longrightarrow \boxed{\tilde{G}(k) = \frac{1}{\omega_0^2 - k^2}}$

PROBLEM: POLES AT $\boxed{\omega_0 = \pm k}$ $\quad G(t) = \int dk\,\tilde{G}(k)\,e^{ikt}$

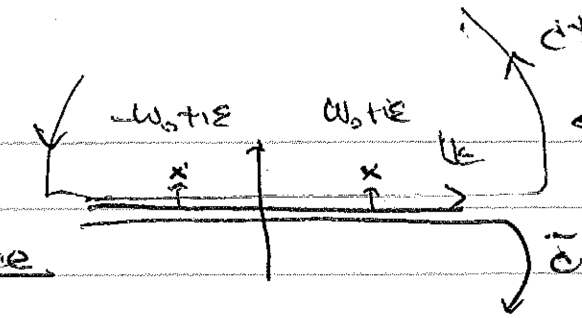
not integrable — inverse fourier transf. not def!

HAVE TO PICK A POLE PRESCRIPTION

$$\frac{1}{\omega_0^2 - k^2} = \frac{1}{\omega_0 + k}\cdot\frac{1}{\omega_0 - k} = \frac{-1}{(k-\omega_0)(k+\omega_0)}$$



LET'S TRY:



← $\tilde{G}$ vanishes @ $\pm i\infty$?

this is a choice.

$$\tilde{G}^{(a)} = \frac{-1}{(k-(\omega_0+i\epsilon))(k-(-\omega_0+i\epsilon))}$$

MINUS sign for inverse transf

$$G^{(a)}(t) = \int_{-\infty}^{\infty} \frac{-dk}{2\pi} \frac{1}{k-(\omega_0+i\epsilon)} \frac{1}{(k-(-\omega_0+i\epsilon))} e^{-ikt}$$

WHEN DOES THIS CONVERGE?

NB: whether FT or FT^{-1} gets e^{+ikx} vs. e^{-ikx}

convergence depends on sign of t!

IF t is + ~~imaginary~~ & k is +imagin., $e^{-ikt} \to e^{\text{positive}}$, not conv!

⇒ have to take lower contour.

→ no poles! ~~$G^{(a)}(t>0)$~~ $\boxed{G^{(a)}(t>0) = 0}$

IF t is − & k is +imag, $e^{-ikt} \to e^{\text{neg}}$, conv.

THEN YOU GET

$$G^{(a)}(t<0) = \oint_{C^+} \frac{-dk}{2\pi} e^{-ikt} \frac{1}{k-(\omega_0+i\epsilon)} \frac{1}{k-(-\omega_0+i\epsilon)}$$

$$= -\frac{1}{2\pi}\left(\mathrm{Res}(\omega_0) + \mathrm{Res}(-\omega_0)\right)$$

$$= -\frac{1}{2\pi}\left(e^{-i\omega_0 t}\frac{1}{-2\omega_0} + e^{i\omega_0 t}\frac{1}{2\omega_0}\right) 2\pi i$$

$$= -\frac{1}{2\omega_0}\left(e^{i\omega_0 t} - e^{-i\omega_0 t}\right)$$

$$= \boxed{-\frac{1}{\omega_0}\sin(\omega_0 t)}$$

← remember this from HW?

$$\rightarrow G^{(a)} = -\frac{1}{\omega_0}\sin(\omega_0 t)\,\Theta(-t)$$

ACAUSAL!!

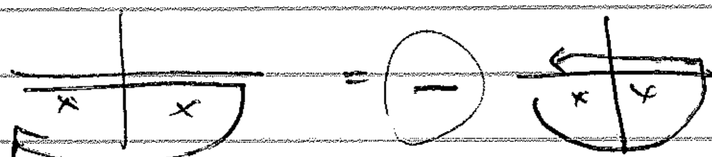
ADVANCED PROPAGATOR

WHAT IF WE'D DEFORMED THE POLES THE OTHER WAY?

CONVERGENCE IS STILL SET BY THE e^{-ikt}, BUT POLES HAVE MOVED: now for

<u>$t>0$</u>: conv. for C^- (LOWER HALF PLANE) + nonzero

<u>$t<0$</u>: conv for C^+, BUT NO POLES → zero.



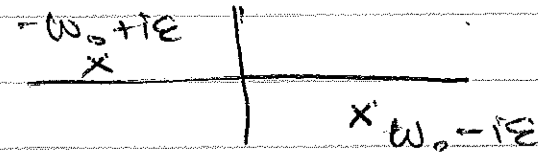
so you pick up a minus sign, too

$$G^{(r)} = +\frac{1}{\omega_0}\sin(\omega_0 t)\,\Theta(t)$$

<u>RETARDED PROPAGATOR</u> - only affected by "causes" in the <u>past</u>.

this is a <u>physical</u> propagator

YET ANOTHER PRESCRIPTION (Feynman)



$$G^{(F)}(t) = \int_{-\infty}^{\infty} \frac{-dk}{2\pi} \frac{1}{k-(\omega_0 - i\epsilon)} \frac{1}{k-(-\omega_0 + i\epsilon)} e^{-ikt}$$

if $t>0$: C^- (LOWER HALF), PICK UP $\omega_0 - i\epsilon$

$$G^{(F)}(t>0) = \frac{-1}{2\pi} \times 2\pi i \frac{1}{2\omega_0} e^{-i\omega_0 t} \underbrace{(-)}_{\text{ORIENTATION}}$$

$$= \boxed{\frac{i}{2\omega_0} e^{-i\omega_0 t}}$$

if $t<0$: C^+ (UPPER HALF), PICK UP $-\omega_0 + i\epsilon$

$$G^{(F)}(t<0) = \frac{-1}{2\pi} 2\pi i \frac{1}{-2\omega_0} e^{i\omega_0 t}$$

$$= \boxed{\frac{-i}{2\omega_0} e^{+i\omega_0 t}}$$

this doesn't look useful ... (IMAGINARY?)
BUT THIS TURNS OUT TO BE USEFUL IN RELATIVISTIC THEORIES: note OPPOSITE SIGN OF ω_0
the $t<0$ SOLUTION IS INTERPRETED AS A "NEGATIVE ENERGY" (ANTI-) PARTICLE MOVING FORWARD IN TIME! [@ this level: this obs. is not obvious & is simply "poetry"]

Sb-60 19.1

SOMETHING FUN: THE Γ FUNCTION

$$\Gamma(z) \equiv \int_0^\infty t^{z-1} e^{-t}\, dt$$

converges for $\mathrm{Re}\, z > 0$

why? OBSERVE: $\frac{d}{dt}\left(t^z e^{-t}\right) = z t^{z-1} e^{-t} - t^z e^{-t}$

$$\Rightarrow \underbrace{\left[t^z e^{-t}\right]_{t=0}^{t=\infty}}_{=0} = z\int_0^\infty t^{z-1} e^{-t}\, dt - \int_0^\infty t^z e^{-t}\, dt$$

$$\Rightarrow \boxed{\Gamma(z+1) = \Gamma(z)}$$

FURTHER $\Gamma(1) = \int_0^\infty t^0 e^{-t}\, dt = 1$

s.t. $\Gamma(n) = n \cdot (n-1) \cdot \ldots = n!$

→ SHOWS UP ALL THE TIME IN COMBINATORICS (& HENCE STATISTICAL PHYS.)

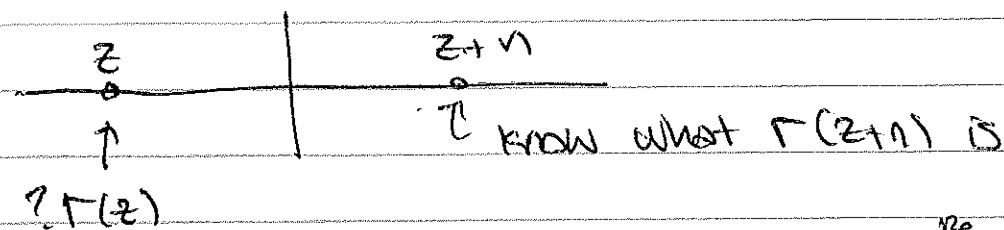
for z more general (inc. neg & im #'s), this is the analytic continuation of the factorial.

POLE STRUCTURE:

$$\left.\begin{aligned} \Gamma(z+1) &= z\Gamma(z) \\ \Gamma(z+2) &= z(z+1)\Gamma(z) \\ \Gamma(z+3) &= z(z+1)(z+2)\Gamma(z) \\ &\vdots \end{aligned}\right\} ?$$

$\Rightarrow \Gamma(z+n) = z(z+1)\cdots(z+n-1)\Gamma(z)$

Use this to extend def of $\Gamma(z)$ to unknown places, eg negative $\mathbb{R}$ axis.

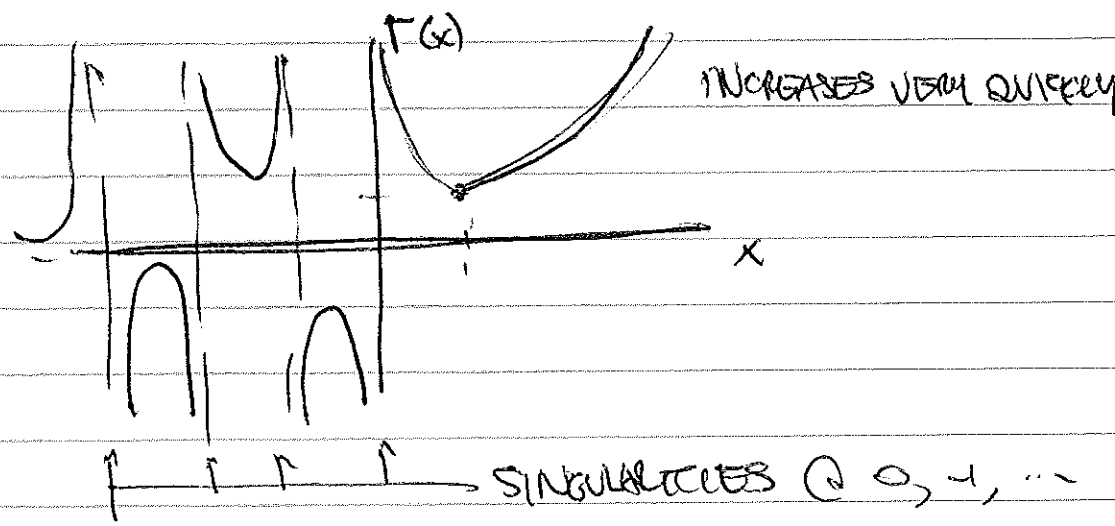


analytic for $\mathrm{Re}(z+n)>0$

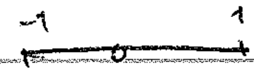
$$\Gamma(z) = \frac{\Gamma(z+n)}{z(z+1)\cdots(z+n-1)}$$

BUNCH OF SIMPLE POLES!

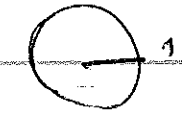
$\Rightarrow \Gamma(z)$ HAS POLES AT $0, -1, -2, \ldots, -n$
w/ RESIDUE $(-1)^n/n!$



INTERESTING APPLICATION: VOLUME OF UNIT BALL OF RADIUS = 1 IN n DIMENSIONS.

eg. 

$V_1 = 2$



$V_2 = \pi R^2$

1. GAUSSIAN INTEGRAL: $\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,dx = I$

TRICK: $I^2 = \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+y^2)}\,dx\,dy$

$$= \int e^{-\frac{1}{2}r^2}\, r\,dr\,d\theta \qquad u = \tfrac{1}{2}r^2 \quad du = r\,dr$$

$$= 2\pi \int_{0}^{\infty} du\, e^{-u}$$

$$= 2\pi \left(-e^{-u}\right)_0^{\infty}$$

$$= 2\pi \qquad \Rightarrow \qquad I = \sqrt{2\pi}$$

2. $\prod_{i=1}^{n} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x_i^2}\,dx_i = (2\pi)^{n/2}$

$$\shortparallel$$

$$\int_{S_{n-1}(r)} \int_0^{\infty} e^{-\frac{1}{2}r^2}\, dA\, dr$$

SURFACE AREA OF $(n-1)$-SPHERE

3. AREA OF $(n-1)$ SPHERE $A_{n-1}(r) = r^{n-1}A(1)$

$$A_{n-1}(1) \underbrace{\int_0^{\infty} e^{-\frac{1}{2}r^2}\, r^{n-1}\,dr}$$

LET $t = \frac{1}{2}r^2 \qquad dt = r\,dr$

$$= \int_0^{\infty} e^{-t}\, t^{\frac{n-2}{2}}\,dt \times 2^{\frac{n-2}{2}}$$

$$= \Gamma(n/2)$$

4. ~~VOLUME~~ $\boxed{A_{n-1}(1) = \frac{2\pi^{n/2}}{\Gamma(n/2)}}$

$$V_n(r) = \int_0^r \underbrace{A_{n-1}(1)\, s^{n-1}}_{A_{n-1}(s)}\, ds$$

$$= \frac{2\pi^{n/2}}{\Gamma(n/2)} \frac{1}{n} r^n = \boxed{\frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} r^n}$$