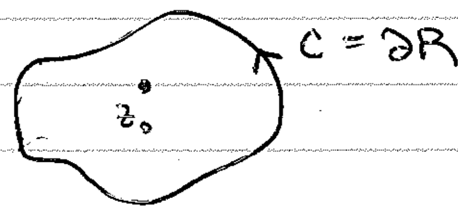
LEC 7: CONTOUR INTEGRALS

LAST TIME: $f = U(x,y) + i\,V(x,y)$

ANALYTIC @ z_0 (& AROUND) $\Leftrightarrow$ CAUCHY-RIEMANN $\Leftrightarrow$ $\exists$ TAYLOR EXP. DERIV. EXIST TO ALL ORDERS

$$U_x = V_y$$
$$U_y = -V_x$$
$$\Updownarrow$$

2D HARMONIC



if f is analytic in R,

$$\boxed{\oint_C f(z)\,dz = 0}$$

CAUCHY THM $\leftrightarrow$ Analytic functions are boring

$$\boxed{f(z) = \frac{1}{2\pi i}\oint_C \frac{f(w)}{w-z}\,dw}$$

CAUCHY INTEGRAL THM

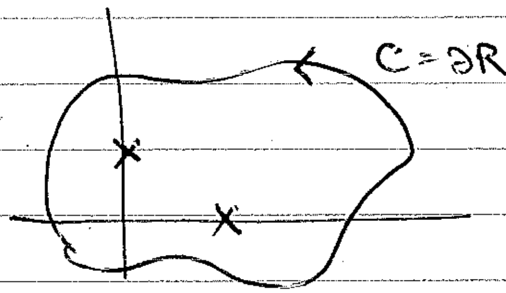
we also talked about branch cuts.
but let's focus on poles.

MEROMORPHIC

OFTEN, A FUNCTION IS ANALYTIC UP TO ISOLATED SINGULARITIES CALLED POLES.

$$\text{eg}\quad f(z) = \frac{1}{(z-1)^2\,(z-i)}$$

POLE OF ORDER 2 @ $z = 1$ POLE OF ORDER 1 @ $z = i$



LAURENT SERIES (about a point z_0) (generalizes TAYLOR)

$$\boxed{f(z) = \underbrace{\sum_{-\infty}^{-1} a_n (z-z_0)^n}_{\text{SINGULAR}} + \underbrace{\sum_{n=0}^{\infty} a_n (z-z_0)^n}_{\text{TAYLOR EXP}}}$$

ASSUME THIS SERIES TERMINATES

(otherwise essential singularity)

a_{-1} : RESIDUE of f @ z_0

LAURENT'S THM

$$a_n = \frac{1}{2\pi i} \oint_C^{\text{enclosing } z_0} \frac{f(z)\,dz}{(z-z_0)^{n+1}}$$

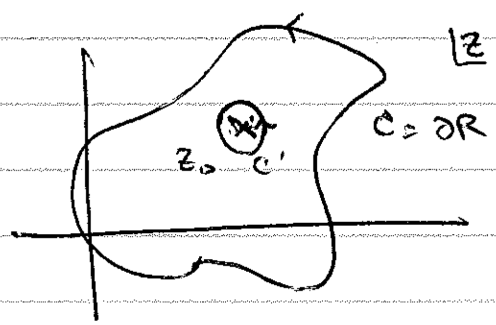
PF: SEE LEC 5 P. 13. ARGUMENT BASED ON CONVERGENCE.

maybe HW EC.

for $n=-1$, the denominator ~~vanishes~~ IS 1

Residue thm: $\Rightarrow$ $\boxed{\oint_C f(z)\,dz = 2\pi i \underbrace{a_{-1}}_{\text{RESIDUE}}}$

~~this is very surprising~~



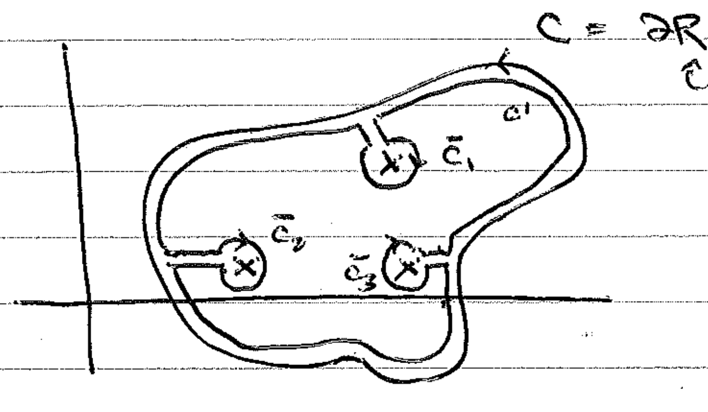
IF NO OTHER POLES IN R, CAN CONTRACT $C \to C'$

MORE GENERALLY:

$$\oint_C f(z)\,dz = 2\pi i \sum \underset{\text{enclosed}}{\text{Res}}$$

& even more generally: mult by # times wound.

RESIDUE THM, PT. 2



c̄ f is MEROMORPHIC IN R; ANALYTIC UP TO POLES

(no branch cuts)

POLES @ z_i w/ CONTOURS C_i GOING AROUND

THE CONTOUR C' is $C + \bar{C}_1 + \bar{C}_2 + \bar{C}_3$

$$= C \underbrace{- C_1 - C_2 - C_3}_{\text{ORIENTATION OF CURVE}}$$

BUT f IS ANALYTIC INSIDE $C' \Rightarrow \oint_{C'} f \, dz = 0$

$$\Rightarrow 0 = \left(\oint_C - \oint_{C_1} - \oint_{C_2} - \oint_{C_3} \right) f \, dz$$

$$\oint_C f \, dz = \sum_i \underbrace{\oint_{C_i} f \, dz}$$

BY RESIDUE THM: these are just the a_{-1}'s IN A LAURENT EXPANSION AROUND EACH z_i

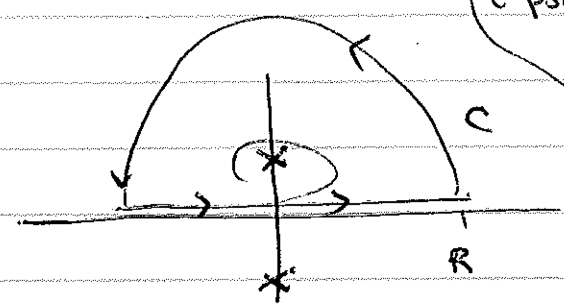
$$= 2\pi i \sum_i \operatorname{Res}(f, z_i)$$

integral over a closed region = sum of residues enclosed.

WHY IS THIS USEFUL?

eg $f(z) = \dfrac{1}{z^2+1} = \dfrac{1}{(z+i)(z-i)}$

POLES @ $\pm i$



$$\oint_C f(z)\,dz = 2\pi i \underbrace{\mathrm{Res}(f, +i)}_{\frac{1}{2i}} = \pi$$

$$\oint C = \underbrace{\int_{-R}^{R} dx}_{\theta = 0} + \underbrace{\int_0^{\pi} d\theta}_{r = R}$$

$$\oint_C f(z)\,dz = \underbrace{\int_{-R}^{R} f(x)\,dx}_{\text{REAL INTEGRAL}} + \underbrace{\int_0^{\pi} f(Re^{i\theta})\, \underbrace{d(Re^{i\theta})}_{Rie^{i\theta}\,d\theta}}_{\int_0^{\pi} \frac{Rie^{i\theta}}{R^2 e^{2i\theta}+1}\,d\theta}$$

OBSERVE: this goes to 0 as $R \to \infty$!

$$\Rightarrow \boxed{\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \pi}$$

← $\mathbb{R}$ integral solved using ANALYTIC PROP on $\mathbb{C}$ PLANE

↳ LOTS OF THINGS COULD BE SAID ABOUT THIS

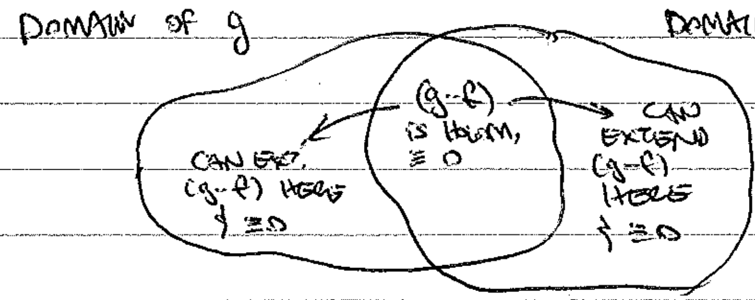
eg WE USUALLY START W/ A $\mathbb{R}$ INTEGRAL W/ PHYSICAL MEANING

WHAT DOES IT MEAN TO GO TO $\mathbb{C}$ INTEGRAL?
eg IS IT UNIQUE? MAYBE $\exists$ A DIFFERENT FUNCTION g THAT AGREES W/ f ON $\mathbb{R}$ LINE, BUT TOTALLY DIFFERENT POLE STRUCTURE $\Longleftrightarrow$ DIFF. CONTOUR INTEGRAL $\Longleftrightarrow$ DIFF ANSWER!

ANSWER: ANALYTICITY IS TOO POWERFUL TO DO THIS! ANALYTIC CONTINUATION

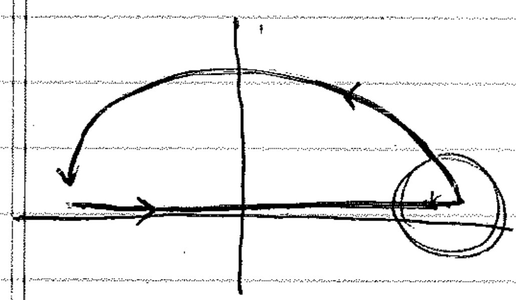
w/in certain conditions (USUALLY SATISFIED) if 2 ANALYTIC FUNCTIONS AGREE ON A DOMAIN, THEY AGREE OVER THEIR COMBINED DOMAINS.

sketch idea (READ ABOUT THIS YOURSELF)



← eg from CAUCHY "AVG VALUE" THM.

ANOTHER CONCERN: EDGE EFFECTS?



$$\int_0^{\pi} \frac{R i e^{i\theta}}{R^2 e^{2i\theta} \boxed{+1}} d\theta$$

ignore this * for now. eg $1/z^2 = f(z)$

$$\sim \frac{i}{Re^{i\theta}}$$

BIG R

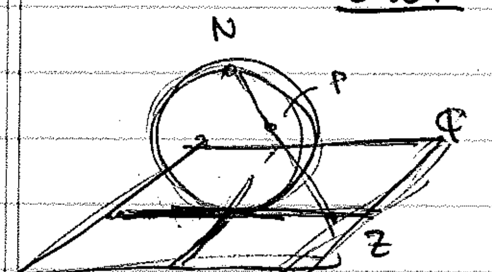
$\exists$ SMALL ENOUGH θ THAT MAY CONTRIBUTE $O(1)$ TO INTEGRAL.

(* ignore issue of pole on int contour for now!)

another way of asking: CONVERGENCE OF LARGE R LIMIT.

ANSWER: WE USUALLY HAND-WAVE THIS AWAY. "take $R \to \infty$ limit first" → rather unsatisfying.

BETTER: RIEMANN SPHERE



MAP EVERY FINITE POINT ON $\mathbb{C}$ TO A POINT ON SPHERE ↔ CARTOGRAPHY

THE POINT ∞, $i\infty$, $a+i\infty$, etc → not on plane ⇔ ie. they correspond to $P \to N$.

IDENTIFY ∞ ($a+ib$) WITH THIS ONE POINT.

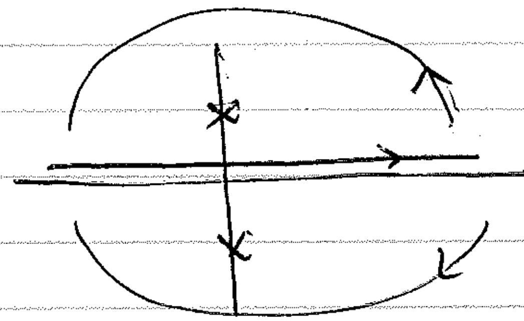
⟶ THEN NO "EDGE" TO SPEAK OF.

⤳ WILL EXPLORE IN HW.

CAHILL P-188

eg. $\underbrace{\displaystyle\int_{-\infty}^{\infty} \frac{2\cos x}{x^2+1}\,dx}$

$$\longrightarrow \frac{e^{iz}+e^{-iz}}{(z+i)(z-i)}$$



WHICH CONTOUR? DEP ON WHICH TERM.

<u>Goal</u>: WANT A CONTOUR ST.

① INCLUDES $\mathbb{R}$ LINE (WHAT WE WANT)

② & HUGE ARC ABOVE/BELOW s.t. INTEGRAL $= 0$.

then $\displaystyle\int_{-\infty}^{\infty} \cdots\, dx + \underbrace{\int_{ARC} \cdots\, dz}_{=0} = \underbrace{\sum 2\pi i \operatorname{Res}(f, z_i)}_{\text{EASY}}$.

$$\frac{e^{iz}\,dz}{\cdots} = \frac{e^{i(R\cos\theta + iR\sin\theta)}\, Ri\, e^{i\theta}\, d\theta}{\underbrace{\cdots}_{\sim R^2}}$$

BUT CONV. IS GOVERNED BY THE EXPONENTIAL

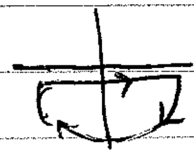
$e^{-R\sin\theta}$ CONVERGES FOR $\sin\theta > 0$

$\rightarrow$ UPPER CONTOUR

$$\oint_{C_U} \overbrace{\frac{e^{iz}}{(z+i)(z-i)}}^{f} = 2\pi i \, \mathrm{Res}(f,i) = 2\pi i \, \frac{e^{-1}}{2i}$$

C_U UPPER HALF $\rightarrow$ POLE ENCL: $z = i$

$$\oint_{\bar{C}_L} \frac{e^{-iz}}{(z+i)(z-i)} = -\oint_{C_L} \cdots = -2\pi i \, \mathrm{Res}(f,-i) = -2\pi i \, \frac{e^{-1}}{-2i}$$



WRONG ORIENT

RIGHT ORIENT.

$$\oint_{C_U} \frac{e^{iz}}{\cdots} + \oint_{\bar{C}_L} \frac{e^{-iz}}{\cdots}$$

$$\rightarrow \int_{-\infty}^{\infty} \frac{2\cos x}{x^2+1}\, dx = \boxed{\frac{2\pi}{e}}$$

REMARK: PRINCIPAL VALUE

WHAT IF YOUR CONTOUR HITS A POLE?
eg. $1/z$
→ PHYSICALLY: ASK WHAT'S HAPPENING.

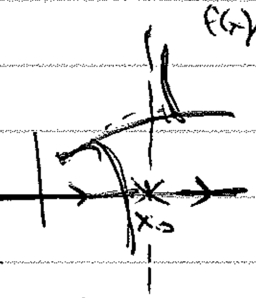
This shows up, eg, w/ VIRTUAL PARTICLES BECOMING REAL.

OLEG WILL TALK ABOUT THIS ON WED.

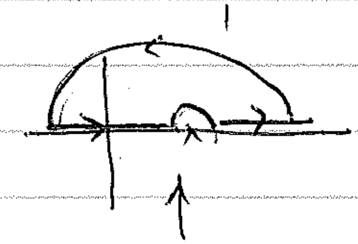
USEFUL IDEA: PRINCIPAL VALUE

M&W p.455
Butkov p.185

$$\mathcal{P}\int_a^b \frac{f(x)}{x-x_0}\,dx = \int_a^{x_0-\varepsilon} \frac{f(x)}{x-x_0}\,dx + \int_{x_0+\varepsilon}^{b} \frac{f(x)}{x-x_0}\,dx$$



IDEA: SINGULAR PARTS CANCEL AROUND x_0

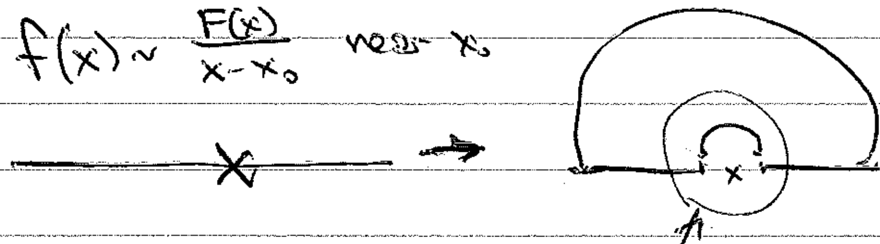


IN CLOSED CONTOUR INTEGRAL → CONTRIBUTES ½ RESIDUE.

~~[illegible]~~

So:

$$f(x) \sim \underbrace{\frac{F(x)}{x - x_0}}_{F \text{ well behaved}} \quad \text{near } x_0$$

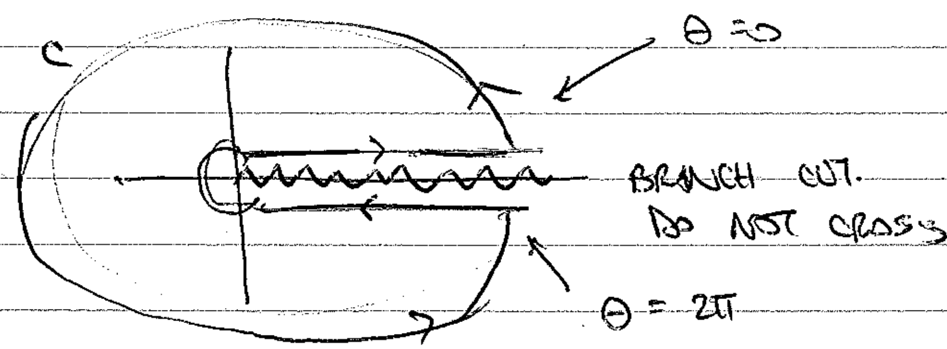


$$\int_{\text{ARC}} \frac{F(z_0)}{z - z_0}\, dz = \int \frac{F(z_0)}{Re^{i\theta}} Rie^{i\theta}\, d\theta$$

$$= \underbrace{\pi i F(z_0)}_{2\pi \left(\frac{1}{2} \operatorname{Res}(f, z_0)\right)}$$

WHAT ABOUT BRANCH CUTS?

$$\int_0^\infty x^{1/3} \underbrace{F(x)}_{\text{well behaved}}\, dx \qquad \text{SPECIFY: } 0 \le \theta < 2\pi$$



$$\oint_C z^{1/3} F(z) = \int_0^\infty F(x)\, x^{1/3}\, dx + \int_\infty^0 x^{1/3} e^{2\pi i (1/3)} F(x)$$

$$= -2i\, e^{\pi i/3} \sin(\pi/3) \int_0^\infty x^{1/3} F(x)\, dx$$

$$= 2\pi i \sum \operatorname{Res}$$