→ we'll try to slow down a bit | M: HBY W: GREG

LECTURE 6: ~~CONTOUR INTEGRALS~~ GREEN'S FUNC. REVIEW ← THW   5 OCTOBER

→ NEW VERS ONLINE

HOMEWORK HINTS

- WHEN CONFUSED: try explicit case, eg $N=3$
- PROBLEMS 1 & 2

WHY DO THESE? UNDERSTAND CALCULUS AS LIN. ALG.

ALSO: GIVES A COMPLEMENTARY LANGUAGE TO UNDERSTAND.

$$\frac{d^2}{dx^2}\Big|_i \sim f_{i+1} - 2f_i + f_{i-1}$$

SO CAN IMAGINE WHAT THIS LOOKS LIKE OVER A BIG MATRIX

$$\begin{pmatrix} \ddots & & & & \\ & -2 & 1 & & 0 \\ & 1 & -2 & 1 & \\ & & 1 & -2 & 1 \\ & 0 & & 1 & -2 \\ & & & & & \ddots \end{pmatrix} \begin{pmatrix} \vdots \\ f_{i-1} \\ f_i \\ f_{i+1} \\ \vdots \end{pmatrix}$$

BOUNDARY CONDITIONS CAN CHANGE THIS!

1a & 2a
$$\begin{pmatrix} -2 & 1 & 0 & \cdots \\ 1 & -2 & 1 & \\ & \vdots & & \end{pmatrix} \leftarrow (-2f_1 + f_2) = f_2 - 2f_1 + f_0 \quad \text{w/ } f_0 = 0$$

try to interp. as "1 −2 1" w/ condition

COMPARE TO 1c

$$\begin{pmatrix} 0 & 0 & 0 & \\ 1 & -2 & 1 & 0 \\ & 1 & -2 & 1 \end{pmatrix}$$

← WHILE 1st ELEM DOESN'T MATTER the (1, −2, 1) is PRESERVED → no BC

BUT $f_1 \neq 0$ necessarily!

on the topic of 1c:

WE ESTABLISHED THAT 1c $\leftrightarrow$ d^2/dx^2 w/ no BC

$\hookrightarrow$ what is the sol to $f''(x) = 0$ w/ no boundary condition?

$$f(x) = ax + b$$ $\leftarrow$ ~~SPANNED BY 2~~ 2 SOLUTIONS

WHAT IS NULL SPACE OF THE MATRIX REP OF d^2/dx^2 ?

$f_{i+1} - 2f_i + f_{i-1} = 0$

RECURRENCE RELATION W/ CONST COEF.

SO ONE GUESS IS $f_n = (\text{const})^n \equiv \alpha^n$

[from function point of view... kind of weird]

$$\alpha^{n+1} - 2\alpha^n + \alpha^{n-1} = \alpha^{n-1}(\alpha^2 - 2\alpha + 1)$$
$$= \alpha^{n-1}(\alpha - 1)^2 = 0$$

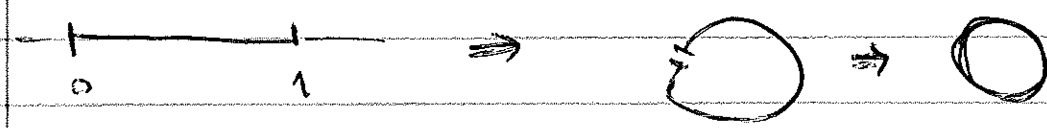
$\Rightarrow$ $\boxed{\alpha = 1}$ $\boxed{f_i = 1}$

that's just one sol. $f(x) = b$

RATHER THAN USING RECURRENCE WE CAN GUESS THE OTHER SOLUTION: $\boxed{f_n = n}$ $\leftrightarrow$ $f(x) = ax$

OTHER BC WE CONSIDERED IS 1D: PERIODIC.

WHAT DOES THIS MEAN?

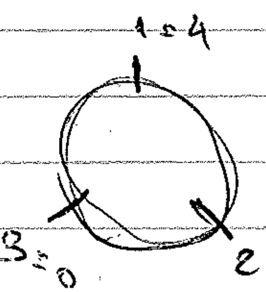


NO BOUNDARY

EXPECT: the $\left(\frac{d}{dx}\right)^2\Big|_i \sim f_{i+1} - 2f_i + f_{i-1}$ structure should carry over.

$$\begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & 1 & \\ & 1 & -2 & 1 & \\ & \boxed{1} & 1 & -2 & \boxed{1} \\ & & & 1 & -2 \end{pmatrix}\begin{pmatrix} f_0\\ f_1\\ f_2\\ f_3\\ f_4\end{pmatrix}$$

$$\begin{pmatrix} \vdots \\ f_0 - 2f_1 + f_2 \\ f_1 - 2f_2 + f_3 \\ f_2 - 2f_3 + \boxed{f_4} \\ \vdots \end{pmatrix} = \begin{pmatrix} f_3 - 2f_1 + f_2 \\ f_2 - 2f_3 + \boxed{f_1} \end{pmatrix}$$

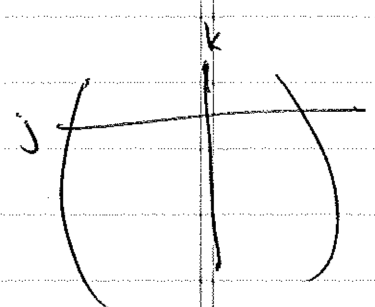


2b) HOW TO VERIFY (T^{-1})?

$$\sum_j (T^{-1})_{ij} T_{jk} = \delta_{ik}$$

given

MUST REPRODUCE "1 −2 1" PATTERN ALONG DIAGONAL



$$T_{jk} = \delta_{k(j-1)} - 2\delta_{jk} + \delta_{k(j+1)}$$

HAVE TO BE CAREFUL @ EDGES.

2a) BC? T_1 is DIRICHLET (by 1a)

$$T_2: \begin{pmatrix} & & \vdots \\ -1 & 2 & -1 \\ \cdots & -1 & 1 \end{pmatrix} \begin{pmatrix} \vdots \\ f_{k-1} \\ f_k \end{pmatrix} = \begin{pmatrix} \vdots \\ -f_k + 2f_{k-1} - f_{k-2} \\ (f_k - f_{k-1}) \end{pmatrix}$$

$(-)\,d^2/dx^2$

WHAT TO MAKE OF THIS?
TRY TO INTERPRET AS $-(1 \ -2 \ 1)$
↳ expect $(-f_{k-1} \quad \underbrace{2f_k \quad -f_{k+1}}_{1})$

gives above value
when $\boxed{f_{k+1} = f_k}$

$f_{k+1} - f_k = 0$ ⟷

NEUMANN!

(2c) ASKS TO COMPARE TO CONTINUUM GREEN'S FUNCTIONS

$\hookrightarrow -(d/dx)^2$ w/ DD or DN

TWO IMPLICIT QUESTIONS

i) ~~[illegible]~~ HOW TO FIND GREEN'S FUNC

ii) WHAT DO THEY LOOK LIKE?

(i) ⇒ follow "direct approach" of PROB 5

(ii) WHAT SHOULD IT LOOK LIKE?

POINTY? SMOOTH? SYMMETRIC?

$$\underbrace{-\frac{d^2}{dx^2} G(x,y)}_{\text{smooth, nice-looking}} = \delta(x-y)$$

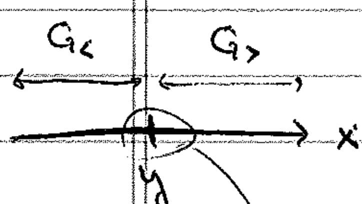
2ND O EQ NEED 2 BC.

spikey, mean-looking → BUT VERY LOCAL.

$$G(x,y) = \begin{cases} G_<(x,y) & \text{if } x<y \\ G_>(x,y) & \text{if } x>y \end{cases}$$

FUNC of x, FIX y! OF COURSE, $0 \le y \le 1$.

s.t. $G_<'' = 0$, $G_>'' = 0$ } 2 × 2ND O EQ NEED 4 BC.



2 MORE BC. FROM WHERE?

NEW BOUNDARY

BOUNDARY @ $X=y$: DEAL WITH $\delta(x-y)$ THE ONLY WAY WE KNOW HOW: INTEGRATE IT TO OBLIVIAN

$$-\int_{y-\epsilon}^{y+\epsilon}\left(\frac{d^2}{dx^2}G\right)dx = \int_{y-\varepsilon}^{y+\varepsilon}\delta(x-y)\,dx$$

$\varepsilon \to 0$

$$-\frac{d}{dx}G\Big|_{y-\varepsilon}^{y+\varepsilon} = 1$$

$' = d/dx$

$$\boxed{G'_>(y,y) - G'_<(y,y) = -1}$$

JUMP DISCONTINUITY

first derivative of Green's function is discontinuous

INTEGRATE AGAIN!

$$\int_{y-\varepsilon}^{y+\varepsilon}\frac{d}{dx}G = -\int_{y-\varepsilon}^{y+\varepsilon}dx$$

$\to 0$! no JUMP

$\Rightarrow \frac{d}{dx}G$ ~~[illegible]~~ $= 0$ @ y

$\Rightarrow$ G IS CONTINUOUS @ y.

$$\boxed{G_>(y,y) = G_<(y,y)}$$

SO THERE ARE YOUR BC-

$G_<(0) = 0$

$G_<(x,y) = ax + b \Longrightarrow ax$

$G_>(x,y) = cx + \cancel{d} \Longrightarrow cx - c$

$G_>(1) = 0$

<u>JUMP CONDITION</u>: $G'_> - G'_< \big|_{x=y} = -1$

$$\boxed{c - a = -1}$$

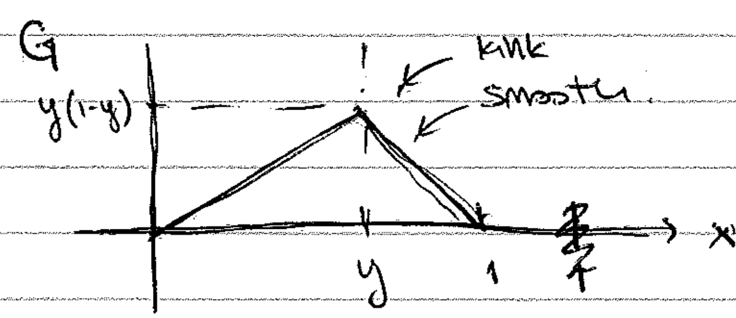
<u>CONTINUITY</u>: $G_> \big|_{x=y} = G_< \big|_{x=y}$

$$\boxed{ay = cy - c}$$

$$\to \underbrace{(c-a)}_{-1} y = c \Rightarrow \boxed{c = -y}$$

$$\Rightarrow \boxed{a = 1 - y}$$

$$\boxed{G(x,y) = \begin{cases} (1-y)x & \text{if } x < y \\ y(1-x) & \text{if } y > x \end{cases}}$$

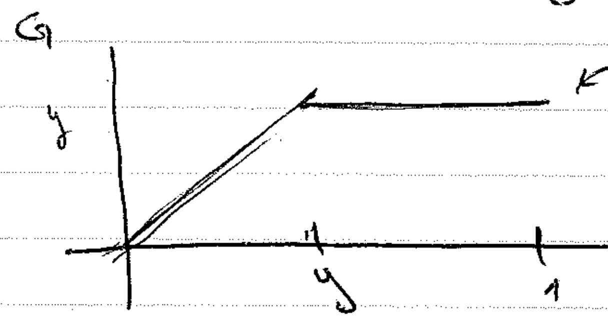


NEUMANN @ $x=1$: SAME AS PREV PAGE,
BUT $G'_>(1, y) = 0 \Rightarrow \boxed{c=0}$ d = ARB.
& $G_> = d$

JUMP: $G'_> - G'_< \big|_{x=y} = -1$
$0 - a = -1 \Rightarrow \boxed{a=1}$

CONTINUITY: $ay = d \Rightarrow \boxed{d=y}$

$$G = \begin{cases} x & \text{if } x<y \\ y & \text{if } x>y \end{cases} = \min(x,y)$$



← indeed, $G'(1)=0$

NOW COMPARE TO DISCRETIZED VERSION

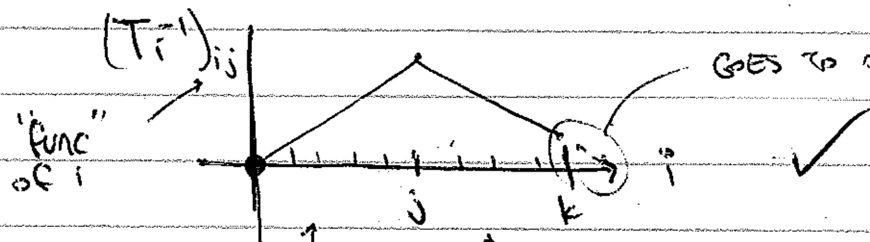
$G \longleftrightarrow T^{-1}$ $\qquad (T^{-1})_{ij}$

$x \longrightarrow i$ } RIGHT?

$y \longrightarrow j$

$\delta(x-y) \longrightarrow |e_j\rangle \longleftarrow \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{pmatrix} \leftarrow j^{th}$ } i GOES OVER THIS RANGE.

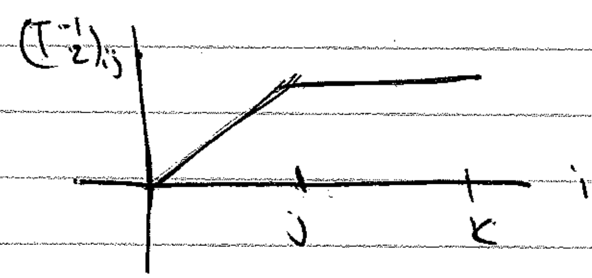
DIRICHL: $(T_1^{-1})_{ij} = \min(i,j) - \dfrac{ij}{k+1} = \dfrac{\min(i,j)(k+1) - ij}{k+1}$



$\dfrac{i(k+1-j)}{k+1}$ $\qquad \dfrac{j(k+1-i)}{k+1}$

inc. w/ i $\qquad$ dec. w/ i

NEUMANN: $(T_2^{-1})_{ij} = \min(i,j)$



① ANALYSIS REMINDER FOR NEXT TIME

$f(z) = u(x,y) + i v(x,y)$

↑ ANALYTIC if func. of z but not $\bar{z}$

⇔ NICE (too nice)

⇔ $\mathbb{C}$ DIFFERENTIABLE

↪ CAUCHY-RIEMANN EQ

$u_x = v_y$ ← $u_x = \frac{\partial}{\partial x} u$, etc.

$v_x = -u_y$

⟶ TAYLOR EXPANSION

Use it in a sentence:

"f is analytic in a region R"

FUNCTIONS MAY BE ANALYTIC IN SOME PLACES, NOT IN OTHERS.

NICE-ENOUGH: MEROMORPHIC

→ ANALYTIC EXCEPT FOR ISOLATED POINTS.

eg. $\dfrac{1}{(z-z_1)(z-i)^3(z-2+3i)^2}$

is MEROMORPHIC: ANALYTIC EVERYWHERE EXCEPT FOR POLES @ z_1

i

$2-3i$

MEROMORPHIC FUNCTIONS HAVE A LAURENT EXP

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-z_0)^n}_{\text{TAYLOR}} + \underbrace{\sum_{m=1}^{\infty} b_m (z-z_0)^{-m}}_{\text{SINGULAR TERMS}}$$

LAURENT EXP w/rt z_0

WE ASSUME THIS SERIES TERMINATES (OR ELSE ESSENTIAL SING.)

Residue @ z_0: b_1

C. THM: $\boxed{\oint_C f(z)\, dz = 0}$ ← ANALYTIC IS "TOO NICE"

↑ CLOSED CURVE AROUND REGION R WHERE f IS ANALYTIC

C. INTEG. FORMULA: $\boxed{\frac{1}{2\pi i} \oint_C \frac{f(w)}{w-z}\, dw = f(z)}$

f(z) is "AVERAGE" OF NEARBY POINTS

⇒ MAX & MIN ARE ON BOUNDARY.

PUNCHLINE FOR NEXT TIME: RESIDUE THM

NOW SUPPOSE f IS "JUST" MEROMORPHIC (has poles)

becomes

$$\boxed{\oint_C f(z)\, dz = \sum_{\substack{\text{POLES}\\ \text{ENC.}}} 2\pi i \, \text{Res}_{\text{POLE}}.}$$