LECTURE 5: ℂ ANALYSIS

23 OCTOBER

REFS:

- Stone & Goldbart — RATHER ADVANCED DISCUSSION
  - CH. 5: ANALYTIC PROPERTIES OF GREEN'S F/NS
  - CH. 17: ℂ ANALYSIS
  - CH. 18: APPLICATIONS INC. PLEMELJ (P. 678), KK
    JORDAN'S LEMMA P. 672

- CAHILL:
  - CH. 5: LOTS OF EXAMPLES, KIND OF SHALLOW?
    INC. PRINCIPAL VALUE §5.18

- P. OLVER, U. MINN. ℂ ANALYSIS & CONF MAPPING
  www.math.umn.edu/~olver/ln_/cml.pdf
  CONF MAP → WHY AIRPLANES FLY

- BYRON & FULLER: *Mathematics of Classical & Quantum Physics*
  - see §6.5: HILB / CAUCHY PRINC VAL.
    §6.6 INTRO TO DISPERSION ← really great!

- TU Graz notes on Adv. Solid State Phys.
  causality & Kramers-Kronig relations
  http://lampx.tugraz.at/~hadley/ss2/linearresponse/causality.php

NOTE: DIS SEC IF YOU WANT MORE PDE DISC!

COMPLEX ANALYSIS

IMMEDIATE GOAL: CONTOUR INTEGRALS
→ for solving (eg) Green's f'ns

if you get nothing else from this unit, then learn to do these!

OTHER IMPORTANT THINGS

"nice" ⇝ analyticity is important in physics
TO BE DEFINED

→ eg CAUSALITY IN DISPERSION RELATIONS
eg. behavior of [scattering] amplitudes where ~~quan~~ virtual/quantum becomes real
eg. WHEN A THEORY BREAKS DOWN

also: SHORTCUT FOR HARMONIC FUNCTIONS IN 2D; ~~eg~~ CONF MAPPING applications to ELECTROSTATICS & FLUID DYNAMICS
↑ why do airplanes fly?

WHY DO $\mathbb{C}$ # SHOW UP IN THE $\mathbb{R}$ WORLD?

1. $\mathbb{R}$ PHENOMENA (LIKE WAVES) WHOSE ANALYTIC PROPERTIES COME IN HANDY
2. PHYSICS THAT IS INHERENTLY $\mathbb{C}$, eg QUANTUM

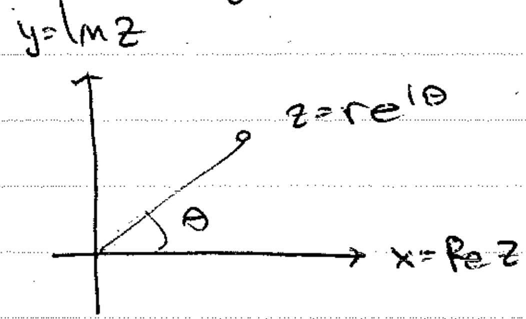
COMPLEX VARIABLES: $z = x + iy$

$\bar{z} = x - iy$

$\uparrow$ or z^*

this is really a 2D vector space WITH ADDITIONAL STRUCTURE
(MULTIPLICATION: $V \times V \rightarrow V$)

$$\begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} \quad \begin{pmatrix} xx' - yy' \\ xy' + yx' \end{pmatrix}$$



COMPLEX FUNCTIONS

in general: $f(z, \bar{z}) \leftarrow$ gives generic function of two variables. may as well have written $f(x, y)$

SPECIAL CLASS OF FUNCTIONS:

ANALYTIC AT z_0 : $f(z)$ w/ DERIV. DEFINED ... in some neighborhood around z_0

$\updownarrow$

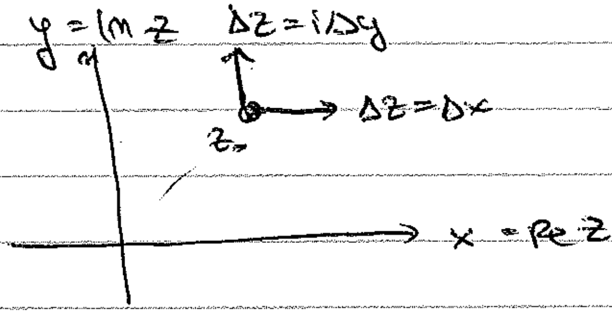
nice $\leftarrow$ type of thing you see in physics: sufficiently differentiable, smooth, etc.

WRITE: $f = \underbrace{u + iv}_{\mathbb{R}\text{ functions}}$

$\mathbb{C}$ DIFFERENTIABLE: $\frac{df}{dz} = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z}$ INDEP. OF Δz

THIS IS A MULTIDIM. SPACE SO YOU CAN TAKE INFINITESIMALS IN DIFF. DIRECTIONS ("DIRECTIONAL DERIVATIVE")

SUFFICIENT TO CONSIDER 2 BASIS DIRECTIONS:



OR IN LANGUAGE OF TANGENT SPACE @ z_0 (later!)

$\boxed{\frac{\partial}{\partial x}, \frac{\partial}{i\partial y}}$ BASIS OF DIFFERENTIAL OPERATORS ON THIS SPACE

I feel that these should be partials until we've established analyticity...

$$\left.\frac{df}{dz}\right|_{dz=dx} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\left.\frac{df}{dz}\right|_{dz=dy} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$\mathrm{Re}\left(\left.\frac{df}{dz}\right|_{dz=dx}\right) = \mathrm{Re}\left(\left.\frac{df}{dz}\right|_{dz=dy}\right)$ & SAME FOR IM

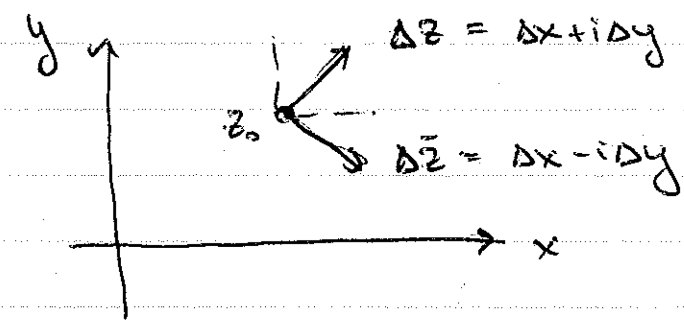
$$\Rightarrow \boxed{\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y} \end{aligned}}$$

CAUCHY-RIEMANN

NB: in ~~plane~~ ordinary $\mathbb{R}^2$, no reason why dir. deriv. has to match coming from diff. directions.

$\mathbb{C}$ DIFFERENTIABLE $\Leftrightarrow$ CAUCHY RIEMANN $\Leftrightarrow$ $f(z,\bar{z}) = f(z)$

COULD IMAGINE A DIFFERENT BASIS FOR THE TANGENT SPACE @ z_0



$$\frac{df}{dz} = \frac{\partial f}{\partial x} + \frac{\partial f}{i\,\partial y} = \Big(\underbrace{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}}_{\text{same}}\Big) + i\Big(\underbrace{\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}}_{\text{same}}\Big)$$

$$\frac{df}{d\bar{z}} = \frac{\partial f}{\partial x} + \frac{\partial f}{-i\,\partial y} = \Big(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\Big) + i\Big(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\Big)$$

PRECISELY WHAT VANISHES UNDER CAUCHY-RIEMANN. $\square$

So: AN ANALYTIC FUNCTION [@ z_0] IS $\mathbb{C}$ DIFFERENTIABLE [IN SOME nbd of z_0] — deriv are "nonlocal"

$\Leftrightarrow$ CAUCHY-RIEMANN

$\Leftrightarrow$ $\boxed{\text{FUNCTION OF } z, \text{ not } \bar{z}}$ — hybrid of 2D & 1D.

nb: of course: differentiable still means something: has to be finite @ z_0, etc.

<u>CAUCHY-RIEMANN</u> $\longleftrightarrow$ u & v are 2D <u>HARMONIC</u>

i.e. $\Delta u = \Delta v = 0$

pf/ $\Delta u = \partial_x^2 u + \partial_y^2 v$

$= \partial_x(\partial_y v) + \partial_y(-\partial_x v)$

$= 0$

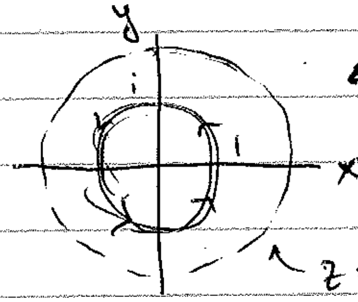
↑ these derivatives commute on flat sp

SO: $\mathbb{C}$ ANALYSIS (& ANALYTIC FUNCTIONS) ARE A SHORTCUT FOR ... 2D ELECTROSTATICS, FLUID FLOW, ...

Byron & Fuller

<u>$\mathbb{C}$ functions as maps of $\mathbb{C} \to \mathbb{C}$</u>

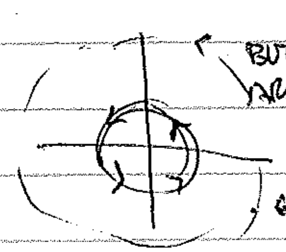
LET'S CONSIDER A SIMPLE DOMAIN/PRE-IMAGE



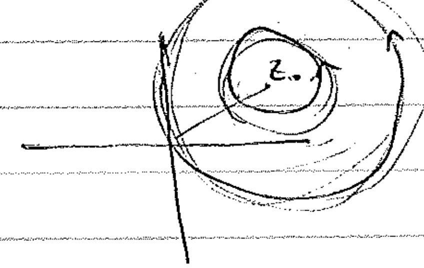
← $z = e^{i\theta}$

↖ $z = r_0 e^{i\theta}$ w/ $r_0 > 1$

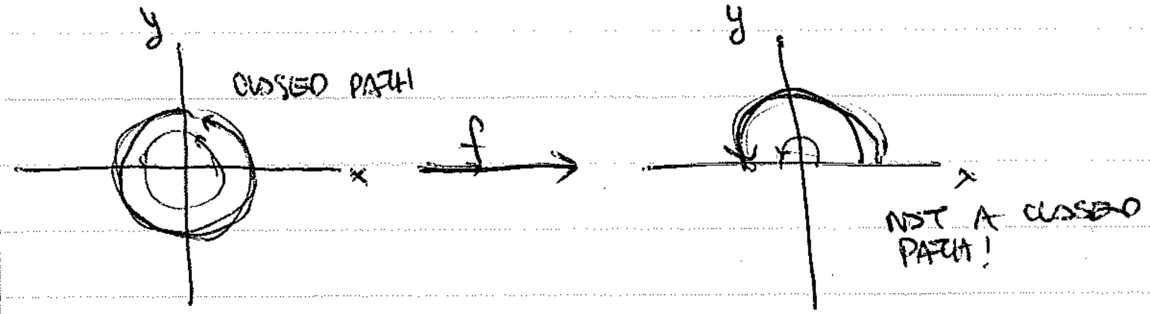
CONSIDER: $f(z) = z^2$



$f = z^2 + z_0$ ↙ SHIFT



WHAT ABOUT $f(z) = \sqrt{z}$?

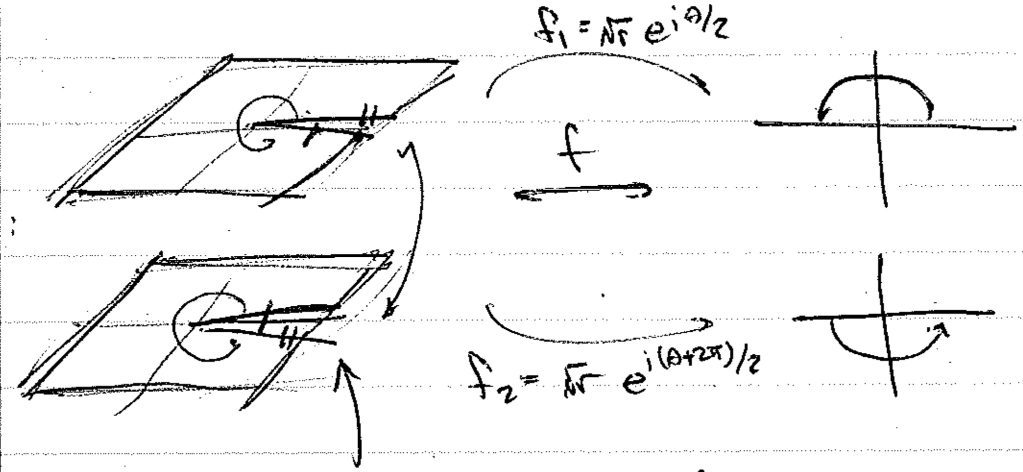


HOW DO WE GET LOWER HALF PLANE IN IMAGE?

WELL... θ GOES FROM $2\pi \to 4\pi$?

$\longrightarrow$ MULTIVALUED!

BETTER: DEFINE DOMAIN W/RT RIEMANN SHEETS



this cut is called a branch cut.

COULD HAVE CUT ON ANY θ.

but not globally

note: $f(z)$ IS ANALYTIC! (everywhere!)

CUT ONLY SHOWS UP AS A GLOBAL PROPERTY IF YOU WIND AROUND THE ORIGIN.

$\checkmark$ $\left.\begin{array}{l} \log = \mathbb{C} \\ \ln = \mathbb{R} \end{array}\right\}$ Byron & Fuller not.

<u>Logarithm</u>: $\log z = \ln r + i\theta$

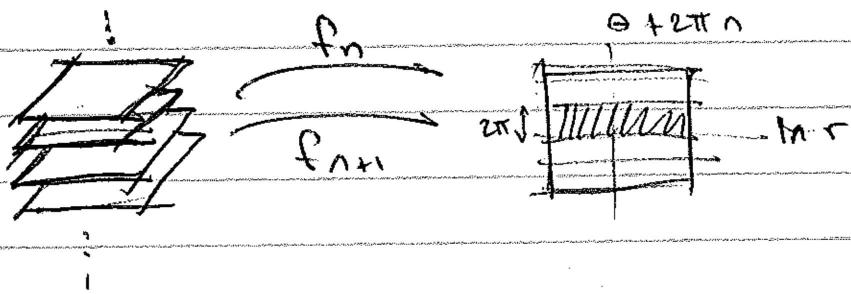
$$\log z_1 z_2 = \ln(r_1 r_2) + i(\theta_1 + \theta_2)$$
$$= \ln(r_1) + \ln r_2 + \cdots$$
$$= \log z_1 + \log z_2$$

<u>obs</u>: every time you go around the origin, you must go to new Riemann sheet b/c θ just keeps increasing.

Is $\log z$ analytic?
YES, EVERYWHERE BUT AT $z=0$.

CAN DEFINE IT "PIECEWISE" w.r.t. SINGLE VALUED f'ns:

$$f_n(z) = \ln r + i\theta + 2\pi n i$$



NOTATION: $\operatorname{Log} z = f_0$ s.t. $e^{\operatorname{Log} z} = z$

So: BRANCH CUTS: DON'T CROSS THEM WHEN YOU INTEGRATE.

ANALYTIC $\leftrightarrow$ NICE

LIFE ISN'T ALWAYS NICE. ⌉ — BUT THAT'S WHEN
PHYSICS ISN'T ALWAYS NICE. ⌋ THINGS GET INTERESTING!

nb. NATURE IS [ALMOST?] ALWAYS ANALYTIC
... BUT OUR DESCRIPTIONS ARE NOT.
SO NONANALYTICITY IS OUR THEORY TELLING US THAT THERE'S MORE PHYSICS TO DESCRIBE!

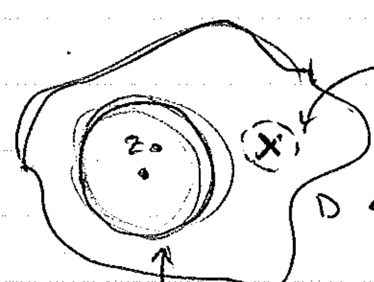
SOMETIMES YOU HAVE SINGULARITIES

eg $\dfrac{1}{(z-1)^2}$

or $\dfrac{1}{i+\sqrt{z}}$ ← here the singularity is only over a single Riemann sheet!

WE'LL MOSTLY ONLY DEAL W/ ISOLATED SINGULARITIES WHERE f IS ANALYTIC EXCEPT AROUND A SMALL NEIGHBORHOOD OF THE SINGULAR POINT.

singularity



D ← f IS ANALYTIC IN REST OF DOMAIN

(DEMY DEF TO ALL θ

$\Rightarrow$ f HAS A TAYLOR SERIES ~~[illegible]~~ THAT CONVERGES AWAY FROM THE SINGULARITY.

Boas

MORE GENERALLY, CAN WRITE LAURENT SERIES

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{m=1}^{\infty} b_m (z-z_0)^{-m}$$

SINGULAR TERMS

if all $b_m = 0$, f is ANALYTIC @ z_0 ($\exists$ A NBD AROUND IT)

if $b_m = 0$ ~~$\forall\, m > \bar{m}$ for sufficiently large m~~, for $m > \bar{m}$, then we say f has a pole of order $\bar{m}$ @ z_0.

if $\bar{m} = 1$, SIMPLE POLE.

if an ∞ number of b's are nonzero, then essential singularity @ z_0.

the coefficient ~~blah~~ b_1 is the RESIDUE of $f(z)$ at z_0.

Key to contour integrals

INTEGRALS

THESE ARE THE USUAL 2D LINE/PATH INTEGRALS.

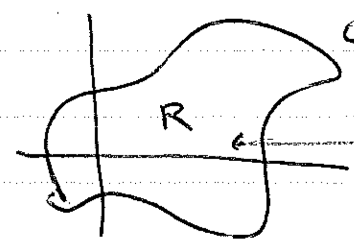
but note $f(z) = u(x,y) + i\,v(x,y)$

WE'RE TREATING THE INTEGRAL AS $\mathbb{R}^2 \to \mathbb{C}$

OR CAN THINK OF THIS AS TWO $\mathbb{R}^2 \to \mathbb{R}$ INTEGRALS.

CAUCHY'S THEOREM IF f IS analytic in R & $C = \partial R$

$$\oint_C f(z)\,dz = 0$$

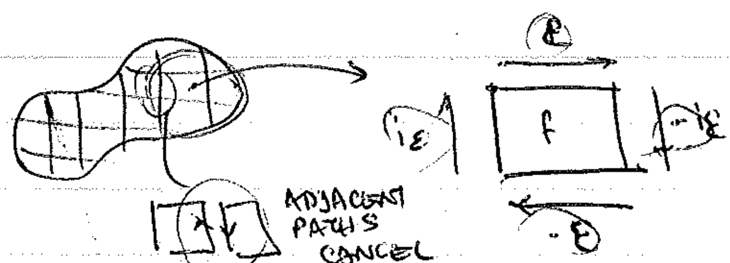


$C = \partial R$ C IS BOUNDARY OF A REGION R

f IS ANALYTIC IN R

Sketch proof: ANALYTIC $\leftrightarrow$ $\exists$ TAYLOR SER & ∞ DIFF. (DIR. DOESN'T MATTER)

then break up R INTO LITTLE REGIONS



EACH (MINI) PATH is $f(z)\,dz \approx f(z_0) + f'(z_0)\varepsilon$

$$\left(f(z_0) + f'(z_0)\varepsilon \cdots\right)\left[\varepsilon - \varepsilon + i\varepsilon - i\varepsilon\right] = 0$$

CONFIRMS OUR INTUITION: IF THINGS ARE TOO NICE, THEN THEY'RE BORING.

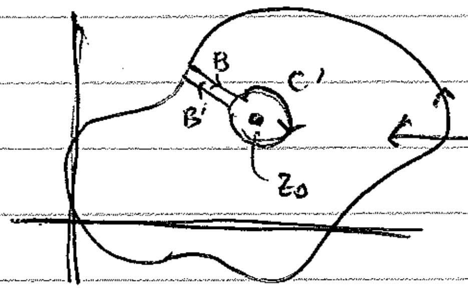
CAUCHY'S INTEGRAL FORMULA : if f is ANALYTIC IN R & $z \in R$

$$f(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)}{z - z_0}\, dz$$

"OBVIOUS" — JUST AVERAGING OVER $f(z)$ IN NBD of z_0

pf. define: $\phi(z) = \dfrac{f(z)}{z - a}$

NOT ANALYTIC @ a.



$\bar{C}' = $ $z_0 + \rho e^{i\theta}$

$$\oint_C \phi(z)\, dz + \oint_{C'} \phi(z)\, dz = 0 \quad \text{by CAUCHY THM}$$

OPP. ORIENTATION

$$\oint_C \phi(z)\, dz = \oint_{\bar{C}'} \phi(z)\, dz$$

SAME ORIENTATION

$$= \oint_{\bar{C}'} \frac{f(z)}{z - z_0}\, dz \qquad z = \rho e^{i\theta} + z_0, \quad dz = i\rho e^{i\theta} d\theta$$

$$= \oint_{\bar{C}'} \frac{f(z_0)}{\cancel{\rho e^{i\theta}}} i\cancel{\rho e^{i\theta}}\, d\theta \qquad \text{take } \rho \to 0$$

$$= 2\pi i \boxed{f(z_0)}$$

$$\boxed{f(z) = \frac{1}{2\pi i} \oint \frac{f(w)}{w - z}\, dw}$$

RESIDUE THEOREM

GO BACK TO PROOF OF CAUCHY'S THM

IF f (WHICH WAS ϕ IN CAUCHY THM PF)
IS SOME otherwize analytic FUNCTION W/ A SIMPLE POLE @ z_0,
THEN:

$$\oint_C f(z)\,dz = \oint_C \underbrace{\sum_n a_n(z-z_0)^n}_{\text{ANALYTIC}\;\to\;0} + \sum_{m=1}^{\infty} b_m (z-z_0)^{-m}$$

contains z_0
f analytic inside C other than z_0

$$\boxed{\oint_C \frac{b_1}{z-z_0}\,dz} + \sum \oint \frac{b_m}{(z-z_0)^m}\,dz \to 0$$

$(z-z_0) \to \rho e^{i\theta}$

Button p. 74
BROWN §61

[Pf. [not for lec] if $f = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$, then $c_n = \frac{1}{2\pi i}\oint_\Gamma \frac{f(z)}{(z-a)^{n+1}}\,dz$]

LAURENT THM

$$\boxed{\oint_C f(z)\,dz = 2\pi i \,\mathrm{Res}(f @ z_0)}$$

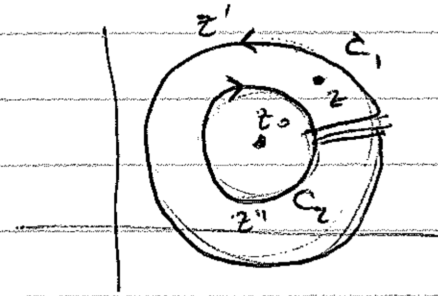
Res @ z_0 is b_1 coeff.

$$= \frac{1}{2\pi i}\oint_C f(z)\,dz$$

$$f = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \qquad a_n = \frac{1}{2\pi i}\oint \frac{f(z)\,dz}{(z-z_0)^{n+1}}$$

ASIDE: Pf of LAURENT THM

CAUCHY:



no poles in annulus

CAUCHY THM: $f(z) = \frac{1}{2\pi i}\oint_{C_1 C_2} \frac{f(z')}{z'-z}\,dz$

$$= \frac{1}{2\pi i}\oint_{C_1} \frac{f(z')}{(z'-z_0)-(z-z_0)}\,dz' + \frac{1}{2\pi i}\oint_{C_2} \frac{f(z'')}{(z-z_0)-(z''-z_0)}\,dz''$$

$$\frac{f(z')}{(z'-z_0)\left[1-\frac{z-z_0}{z'-z_0}\right]} \qquad \frac{f(z'')}{(z-z_0)\left[1-\frac{z''-z_0}{z-z_0}\right]}$$

note: $\left|\frac{z''-z_0}{z-z_0}\right| < 1$ & $\left|\frac{z-z_0}{z'-z_0}\right| < 1$

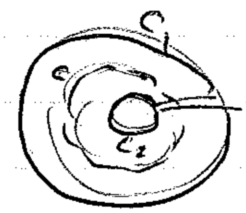
use: $\left(1-\frac{z-z_0}{z'-z_0}\right)^{-1} = \sum_{n=0}^{\infty}\left(\frac{z-z_0}{z'-z_0}\right)^n$

TAYLOR SERIES IN $\left({z-z_0}/{z'-z_0}\right)$

$$= \sum_{n=0}^{\infty} (z-z_0)^n \frac{1}{2\pi i}\oint_{C_1} \frac{f'(z')}{(z'-z_0)^{n+1}}\,dz'$$
$$+ \sum_{m=0}^{\infty} \underbrace{\frac{1}{(z-z_0)^{m+1}}}_{z\text{ dep here.}} \frac{1}{2\pi i}\oint_{C_2} (z''-z_0)^m f(z'')\,dz''$$

ANALYTIC IN ANNULUS

CONSEQUENCES: take $C_1 \to C$, $C_2 \to C$



→ set $m = -n - 1$

$$\Rightarrow \boxed{f(z) = \sum_{n=-\infty}^{\infty} (z-z_0)^n \underbrace{\frac{1}{2\pi i} \oint_C \frac{f(z')}{(z'-z_0)^{n+1}}\, dz'}_{\text{COEFFICIENT OF LAURENT EXPANSION}}}$$

BOAS 14.6

FINDING RESIDUES

① LAURENT SERIES, COEFF OF $\frac{1}{z-1}$ TERM

② SIMPLE POLE

↳ MULTIPLY $f(z)$ by $(z-z_0)$ & take $z \to z_0$.

eg. 1.13 GOOD. →

eg. $f(z) = \frac{z}{(2z+1)(5-z)}$ not $(2z+1)$

Res$(-1/2)$ ↳ $(z+\frac{1}{2}) f(z) \big|_{z=z_0} = \frac{-1/2}{(5+1/2)}$

hw? →

③ MULTIPLE POLES: if POLE OF ORDER n:

- MULTIPLY BY $(z-z_0)^m$ $(m \geq n)$
- DIFFERENTIATE $(m-1)$ TIMES
- DIVIDE BY $(m-1)!$
- EVALUATE @ $z = z_0$.