LEC 4: GREEN'S FUNCTIONS

TODAY: COMPLETENESS OF EIGENF.
BIG PICTURE OF (P/O) DE's.
GREEN'S FUNC IN FINITE DIM → SUPER POS.
GREEN'S FUNC IN DE.
EXAMPLES

REFS: Stone & Goldbart: 4.3, 5.4
Cahill: p. 124, 284
Goertzel ch. 12
Boaz 15.8

QUICK NOTE:
COMPLETENESS OF EIGENFUNCTIONS

$$\hookrightarrow \langle e_i | e_j \rangle_w = \int_0 dx\, w(x) e_i^*(x) e_j(x) = \delta_{ij}$$

ALSO TRUE: $\boxed{\sum_i w(y) e_i^*(y) e_i(x) = \delta(x-y)}$

pf/ $|f\rangle = \sum_i a_i |e_i\rangle \leftarrow e_i(x)$

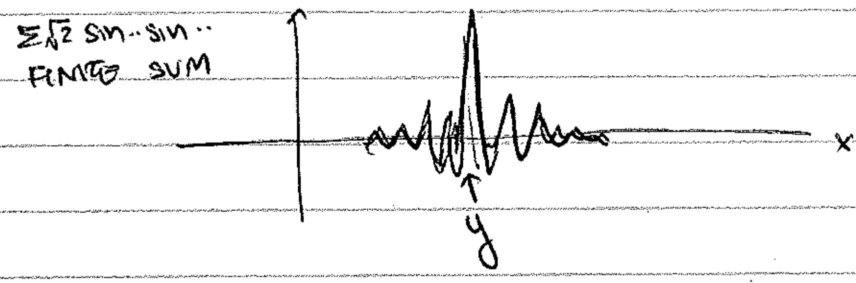
$f(x)$ ↗ $\quad a_i = \langle e_i | f \rangle = \int dy\, w(y) e_i^*(y) f(y)$

$$= \sum_i \int dy\, w(y) e_i^*(y) e_i(x) \quad f(y)$$

WHICH IS PRECISELY THE MEANING OF δ

eg. FOURIER SERIES: $|e_i\rangle = \sqrt{2}\sin(n\pi x)$

$$\sum_{n=1}^{\infty} \sqrt{2}\sin(n\pi x)\sin(n\pi y) = \delta(x-y)$$



<u>Remarks</u>: this relation is not formally convergent, not surprising, $\delta(x-y)$ is not formally a function.

OK. REMEMBER THIS RESULT.

<u>BIG PICTURE</u>

ONE INDEP VAR. → ODE $\sim y'(t) = \ldots$

not nec integrable! → PDE

ODE / PDE → <u>LINEAR</u> → HOMOGENEOUS → BAG OF TRICKS eg SEP OF VARS

<u>LINEAR</u> → INHOMOGENEOUS → GREEN'S FUNC

ODE / PDE → <u>NONLINEAR</u> → MORE SOPHIST'C BAG OF TRICKS

<u>NONLINEAR</u> → APPROXIMATIONS eg. PERTURBATION THY. → Feynman Diagrams.

$$L \sim \varphi\,\partial^2\varphi + m^2\varphi + \lambda\varphi^4$$

GREEN'S FUNCTIONS:

FIN. DIM. ANALOGY: $A\,|f\rangle = |g\rangle$

$\underbrace{A}_{\text{dynamics}}\ \underbrace{|f\rangle}_{\text{state, eg position}} = \underbrace{|g\rangle}_{\text{source, eg. "kick"}}$

dynamics → EFFECT ← CAUSE ← source

We know what the effect is from physics; @ least infinitesimally

ASSUME INVERTIBLE... $|f\rangle = A^{-1}|g\rangle$

SUPERPOSITION: if $|g\rangle = |g_1\rangle + |g_2\rangle$
then:

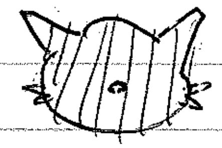
$$|f\rangle = A^{-1}|g_1\rangle + A^{-1}|g_2\rangle$$

$$\rightarrow \sum_i A^{-1} \underbrace{c_i |e_i\rangle}_{|g\rangle}$$

ALL YOU NEED TO KNOW IS EFFECT OF A^{-1} ON BASIS ~~POINT~~ STATES THEN BUILD UP SOLUTION BY PUTTING THEM TOGETHER.

Physically:

eg electrons

 $= \Sigma$ POINT CHARGES

CHARGE CONFIG
OF FINITE EXTENT

the electric potential on LHS
is just the sum of electric potentials
on RHS

good exercise

the electrons ~~are~~ are δ-func's
in charge.
EACH ONE GIVES A $\frac{e}{4\pi r}$
CONTRIBUTION TO Φ.

DIFF EQ for GREEN'S FUNCTION

for n vars.
eg x is n-vec.

$$\mathcal{O}\, G(x) = \delta^{(n)}(x)$$

LIN op. PREVIOUSLY A — GREEN'S — δ fn. in n-DIMENSIONS

$$G(x) \text{ "="}\ \mathcal{O}^{-1}\, \delta^{(n)}(x)$$

MORE PRECISELY:

$$\mathcal{O}_{x'}\, G(x, x') = \delta^{(n)}(x - x')$$

OBS POINT. SOURCE POINT.

HOW IT WORKS eg $\left(\frac{d}{dx}\right)^2 + M^2$

SUPPOSE $\mathcal{O}_x f(x) = g(x)$

SOURCE

THEN WANT: $f(x) = \mathcal{O}_x^{-1} g(x)$

$$\equiv \int \mathcal{O}_x^{-1} \delta(x-y) g(y)\, dy$$

$$= \int G(x,y)\, g(y)\, dy$$

& that's the idea.

Cahill eg 6.42, p. 234

eg. POISSON'S EQ.

PICK COULOMB GAUGE $\nabla \cdot A = 0$

$\rightarrow \quad -\Delta \phi = \sum_i^3 \partial_i^2 \phi = \rho \qquad G = (-\Delta)^{-1}$

GREEN'S FUNCTION IS $\quad -\Delta G = \delta^{(3)}$

$$\phi(x) = \int G(x-y)\, \rho(y)\, dy$$

HOW TO SOLVE? ONE USEFUL TRICK IS TO FOURIER TRANSFORM

$$G(\underline{x}) \equiv \int e^{i\underline{k}\cdot\underline{x}}\, \tilde{G}(\underline{k})\, đ^3k$$

$$\delta^{(3)}(\underline{x}) = \int đ^3k \; e^{i\underline{k}\cdot x} \quad \leftarrow \forall\, x \neq 0, \text{ destr. int.}$$

$$(\, đ = d/2\pi)$$

then: $$-\Delta G(x) = -\Delta \int e^{i\underline{k}\cdot x}\, \tilde{G}(\underline{k})\, đ^3k$$
$$= \int |\underline{k}|^2 e^{i\underline{k}\cdot\underline{x}}\, \tilde{G}(\underline{k})\, đ^3k$$

compare to $\delta^{(3)}(x)$

$$\Rightarrow \tilde{G}(\underline{k}) = 1/|\underline{k}|^2$$

$$\Rightarrow G(\underline{x}) = \int e^{i\underline{k}\cdot\underline{x}} \frac{1}{|\underline{k}|^2}\, đ^3k = \frac{1}{4\pi r}$$

from contour integral (next time)

Arfken p. 888
Stogo p. 154

EIGENFUNCTION EXPANSION

$$\Theta\, e_i(x) = \lambda_i\, e_i(x)$$

eigenf w/ eigval λ_i

eg $\sqrt{2}\sin(n\pi x)$ w/ $\lambda = -n^2$ for $\Theta = (d/dx)^2$

WE SHOWED EARLIER THAT

$$\cancel{\langle e_i | e_j \rangle_w} \quad \sum_i w(y)\, e_i^*(y)\, e_i(x) = \delta(x-y)$$

GREEN'S FUNCTIONS SATISFY

$$\Theta_x G(x,y) = \delta(x-y)$$

$$\boxed{G(x,y) = \sum_i \frac{w(y)\, e_i^*(y)\, e_i(x)}{\lambda_i}}$$

ANALOG OF DIAGONAL MATRIX:

$$A\,|f\rangle = |g\rangle$$

$$\begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{pmatrix} \Rightarrow \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} g_1/\lambda_1 \\ g_2/\lambda_2 \\ \vdots \end{pmatrix}$$

$$A = \sum_i \lambda_i |e_i\rangle\langle e_i|$$

example

IN SPHERICAL COORDINATES THERE ARE EIGENFUNCTIONS OF THE (ANGULAR) PART OF THE LAPLACIAN $(-\Delta)$: THE SPHERICAL HARMONICS

THIS LEADS TO ANOTHER NICE REPRESENTATION OF THE GREEN'S FUNCTION

$$G(\underline{x}-\underline{x}') = \frac{1}{4\pi r} \quad \leftarrow \text{I hand waved this earlier}$$

exercise! $\Rightarrow$

$$= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{l,m}(\theta,\phi) Y^*_{l,m}(\theta',\phi')$$

PARTIAL WAVE (ORBITAL ANG. MOMENTUM) — MOMENT IN z DIR. — $r_> = \max(r, r')$, $r_< = \min(r, r')$

s.t. for source $\rho(x)$, POTENTIAL IS

$$\phi = \sum_l \sum_m \frac{1}{2l+1} \int \frac{r_<^l}{r_>^{l+1}} Y_{lm}(\theta,\phi) Y^*_{lm}(\theta',\phi') \rho(x') \, d^3x'$$

$P_l(\underline{r}\cdot\underline{r}')/4\pi$ (LEGENDRE)

SUPPOSE WE'RE OBSERVING FAR FROM DISTR.

$\rightarrow r > r' \quad \Rightarrow \quad r_> = r, \quad r_< = r'$

$$\phi = \sum_{l=0}^{\infty} \frac{1}{2l+1} \sum_{m=-l}^{l} \frac{Y_{lm}(\theta,\phi)}{r^{l+1}} \underbrace{\int (r')^l Y^*_{lm} \rho \, d^3x'}_{Q_l^m}$$

Q_l^m MULTIPOLE MOMENTS.

BERIZEL
P. 166
§ 12.3

ANOTHER EXAMPLE

$\Theta = -(d/dx)^2 + 1$

ON SPACE $x \in (-\infty, +\infty)$ & BC $f(\pm\infty) = 0$

$$G(x,y) = \Theta^{-1} \underbrace{\int_{-\infty}^{\infty} e^{ik(x-y)}\, dk}_{\delta(x-y)}$$

$$= \int \frac{e^{ik(x-y)}}{k^2+1}\, dk$$

from FOURIER ON LHS
then dividing through.

REMARK: SUPPOSE x HAD DIM. (eg LENGTH)
then $\Theta \to -\ell^2 (d/dx)^2 + 1$
HOW IS $G(x,y)$ REP. ABOVE
MODIFIED?

$$\hookrightarrow \int dk \frac{e^{ik(x-y)}}{\ell^2 k^2 + 1}$$

AS $\ell \to 0$: $G(x,y) \to \delta(x-y)$
MAKES SENSE, $\Theta \to 1$. & $\Theta G = \delta$

CLAIM (WE'LL PROVE LATER):

$$G(x,y) = \frac{1}{2} e^{-|x-y|}$$

Q: WHERE DOES ℓ GO?

$\Theta G = \delta$ ← HAS DIM.

$\downarrow$

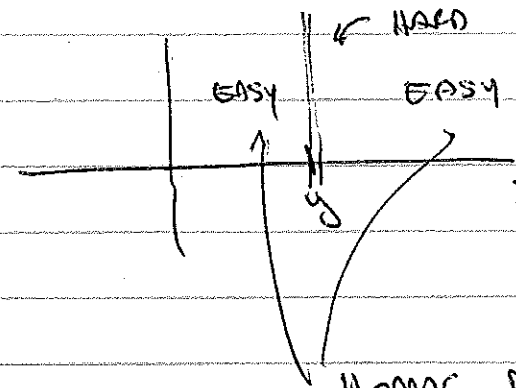
$\frac{1}{2\ell} e^{-|x-y|/\ell}$

SAME EG: SOLUTION FROM PATCHING.

$$\Theta\, G(x,y) = \delta(x-y)$$

↑ $-\left(\frac{d}{dx}\right)^2 + 1$

w/ BC $G(\pm\infty, y) = 0$



$x > y$

↓

$G_>(x,y) = 0$

& sim for $x<y$.

ASSUMING SEPARABLE SOLUTIONS

$$\left.\begin{aligned} G_<(x,y) &= e^{x} a(y) \\ G_>(x,y) &= e^{-x} b(y) \end{aligned}\right\} \text{from BC}$$

$$G(x,y) = G_< \Theta(y-x) + G_> \Theta(x-y)$$

WANT TO PATCH THESE SOLUTIONS TOGETHER

$$\frac{d}{dx} G(x,y) = e^{x} a(y)\, \Theta(y-x) - e^{-x} b(y)\, \Theta(x-y)$$

$$\boxed{- e^{x} a(y)\, \delta(x-y) + e^{-x} b(y)\, \delta(x-y)}$$

↓

MUST VANISH!

OTHERWISE ~~ΘG~~ ~~ΘG~~ ΘG

CONTAINS $[\cdots]\, \frac{d}{dx}\delta(x-y)$

WHICH IS INCOMPATIBLE w/ $\Theta G = \delta$.

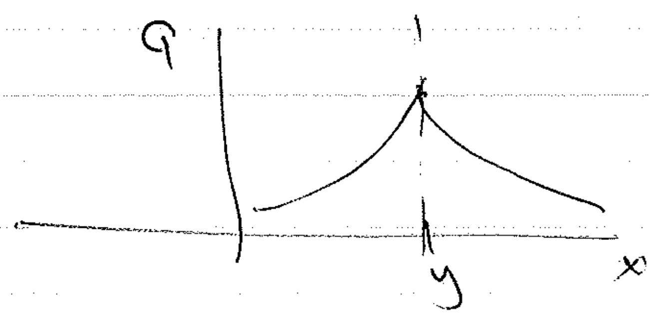
then : $\underbrace{e^{-y} b(y) - e^{y} a(y)}_{\text{TRICK}} = c(y)$

$$\Rightarrow \frac{d}{dx} G(x,y) = c(y) \left[e^{-(y-x)} \theta(y-x) - e^{-(x-y)} \theta(x-y) \right]$$

$$\left(\frac{d}{dx}\right)^2 G(x,y) = G(x,y) - 2\, c(y)\, \delta(x-y)$$

$c(y) = -1/2$
TO SATISFY GREEN'S.

OBSERVE : for 2ND O DE,
$d/dx\, G(x,y)$ IS DISCONTINUOUS .



A

MW p.257 A BETTER EXPLANATION OF EIGENVEC EXP.

$$\Theta f = g$$

$$\uparrow \qquad \nwarrow$$

$$\Sigma c_i e_i \qquad \Sigma d_i e_i \qquad w/ \ \Theta e_i = \lambda_i e_i$$

$$\Rightarrow c_i \lambda_i e_i = d_i e_i \quad \Rightarrow c_i = \frac{d_i}{\lambda_i} = \frac{\langle e_i | g \rangle}{\lambda_i}$$

$$\Rightarrow f = \sum_i c_i e_i = \sum_i \frac{e_i(x)}{\lambda_i} \int w(y) e_i^*(y)\, g(y)\, dy$$

$$= \int \underbrace{\Sigma \frac{e_i(x) e_i^*(y)}{\lambda_i} w(y)}_{G(x,y)} \, g(y)\, dy$$