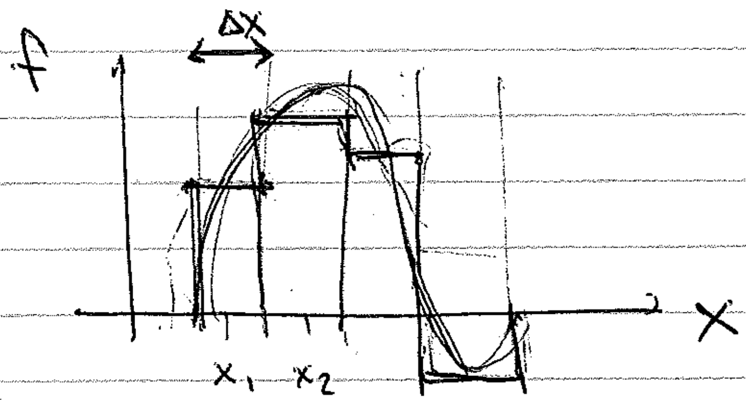
LECTURE 3 23 SEPT

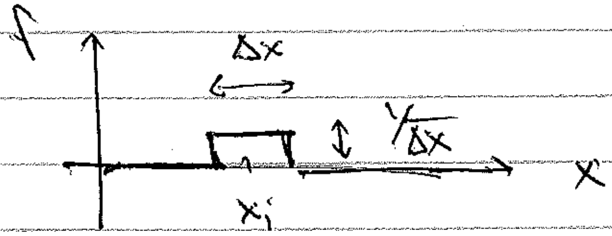
LAST TIME: "DISCRETIZED FUNCTION" AS A CRUTCH TO UNDERSTAND FUNCTION SPACE AS VECTOR SPACE.




BASIS VECTORS: CORRECTION FROM LAST TIME

$$|e_i\rangle = \begin{cases} \frac{1}{\Delta x} & \text{if } x \in [(x_i - \Delta x/2), (x_i + \Delta x/2)) \\ 0 \end{cases}$$

$e_i(x)$




WHY? A GOOD BASIS is ORTHONORMAL

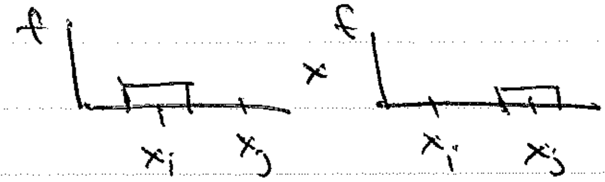
DEPENDS ON INNER PRODUCT.

ON OUR FAVORITE FUNCTION SPACE (HILBERT SPACE)
WE HAVE THE FOLLOWING INNER PRODUCT

$$\langle f|g\rangle = \cancel{\int dx\, f^*(x)\, g(x)}\ \sum_i f(x_i)^* g(x_i)\,\Delta x$$

$$\cancel{\langle e_i|e_i\rangle = \int_{x_i-\Delta x/2}^{x_i+\Delta x/2} \frac{1}{\Delta x}\,\frac{1}{\Delta x} = \frac{1}{\Delta x}}$$ ?!

$\langle e_j|e_i\rangle = 0$ because
$j\neq i$




$= 0$

"has no support"

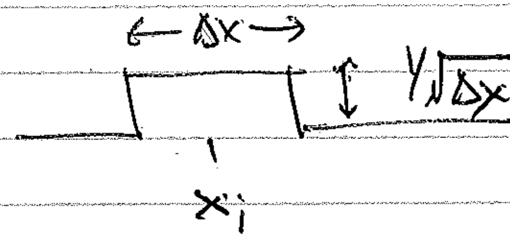
$$\langle e_i|e_i\rangle = \sum_i \left(\frac{1}{\Delta x}\right)\left(\frac{1}{\Delta x}\right)\Delta x = \frac{1}{\Delta x}$$

or: $\langle e_i|e_j\rangle = \frac{1}{\Delta x}\,\delta_{ij} \longrightarrow \delta(x_i - x_j) \neq \delta_{ij}$

SO I LIED LAST TIME. THIS IS A WEIRD BASIS & IS PART OF THE WEIRDNESS ~~ODDNESS~~ OF $\infty$-DIM VECTOR SPACES! IN PART THIS IS BECAUSE $\delta$ FUNCTIONS ARE NOT REAL FUNCTIONS AND AREN'T PART OF OUR FUNCTION SPACE. WE ~~NEED~~ WANT TO USE THEM AS BASIS VECTORS, THE COST IS WEIRD BEHAVIOR LIKE $\langle e_i|e_j\rangle$

nb I DON'T THINK YOU CAN FIX THIS

eg. IF YOU USED

$|e_i\rangle =$ 


then $\langle e_i|e_j\rangle = \sum_k e_i(x_k)\,e_j(x_k)\,\Delta x$

$= \delta_{ij}$ ✓

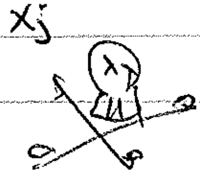
BUT THEN: $e_i(x)$ IS NOT NORMALIZED TO INTEGRATE TO 1

$\rightarrow$ doesn't become $\delta$-fn.

$$|f\rangle = \sum_i \underbrace{f(x_i)\underset{\uparrow\ \text{weird}}{\sqrt{\Delta x}}}_{c_i}\,|e_i\rangle$$

WHAT'S REALLY HAPPENING IS

$$\lim_{x_i \to x_j} \underbrace{\int dx\, \delta(x-x_i)\,\delta(x-x_j)}_{\text{LHS IS DISASTER WHEN } x_i = x_j} = \underbrace{\delta(x_i - x_j)}_{\text{INTERPRET AS DISTRIBUTION}}$$



ANYWAY: ALL THAT IS TO SHOW THAT SOMETIMES OUR INTUITION & OUR SIMPLE MODELS BREAKS DOWN.

ESP WHEN GOING FROM FINITE $\to \infty$ DOF. (eg. tunneling)

AS PHYSICISTS: PUSH ON ANYWAY! (UNLESS IT BREAKS WHERE WE NEED IT.)

eg INFORMATION LOSS PROBLEM

DERIVATIVES

$$f_i = f(x_i) \qquad \frac{d}{dx} f(x_i)$$

$$\begin{array}{c} \qquad\quad i \qquad\quad i+1 \\ \begin{pmatrix} \ddots & & & \\ & \frac{-1}{\Delta x} & \frac{+1}{\Delta x} & \\ & & & \ddots \end{pmatrix} \end{array} \begin{pmatrix} \vdots \\ f_i \\ f_{i+1} \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ \frac{1}{\Delta x}\left(\underbrace{f(x_{i+1})}_{f(x_i+\Delta x)} - f(x_i)\right) \end{pmatrix}$$

So: DIFFERENTIAL OPERATORS ARE SIMPLY MATRICES IN THIS PACE

WHAT ABOUT SECOND DERIVATIVES?

$$\begin{array}{c} \begin{array}{ccc} i-1 & i & i+1 \end{array} \\ \begin{pmatrix} \frac{1}{\Delta x^2} & \frac{-2}{(\Delta x)^2} & \frac{1}{(\Delta x)^2} \end{pmatrix} \end{array} \begin{pmatrix} \vdots \\ f_{i-1} \\ f_i \\ f_{i+1} \\ \vdots \end{pmatrix} = \begin{pmatrix} \dfrac{(f(x_i+\Delta x)-f(x_i))-(f(x_i)-f(x_i-\Delta x))}{\Delta x^2} \end{pmatrix}$$

Q: HAS ANYONE USED 3RD O DIFF EQ?

$$\boxed{d^2f/dx_i^2}$$

<u>OBSERVE</u>: as you expect intuitively from taylor expansion,

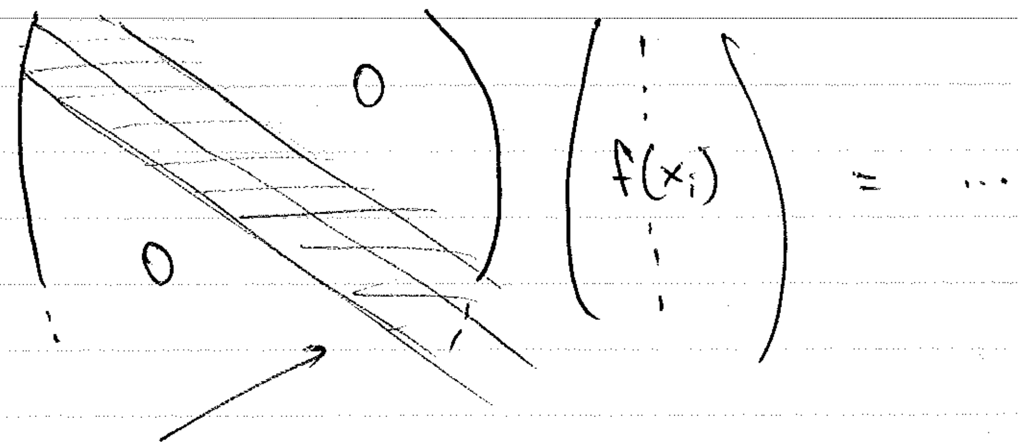
HIGHER ORDER DERIVATIVES ARE INCREASINGLY <u>NONLOCAL</u>, PROBE SPACE/TIME/SPACETIME POINTS THAT ARE FAR APART

You also probably have a good sense now that Physics is <u>LOCAL</u>.

↳ SO IT IS DESCRIBED BY LOW ORDERS OF DERIVATIVES.

NEED <u>SOME</u> DERIV. B/C WE'RE DESCRIBING <u>DYNAMICS</u> (how things change in time)

SO PHYSICS LOOKS SOMETHING LIKE THIS:




typically up to <u>2ND ORDER</u>.

WHY SECOND ORDER ? NOT THIRD? FOURTH?

↳ DIMENSIONAL ANALYSIS.

$\frac{d}{dt}$ HAS DIMENSIONS.

<u>ACTION</u> DOES TOO (not in natural units)

in nat units $e^{iS/\hbar}$ I GUESS.

$$S = \int dt\, L = \int \underbrace{d^4x}_{\text{DIM}}\, \underbrace{\mathcal{L}}_{\text{DIM}}$$

IF YOU WANT A TERM IN $\mathcal{L}$ WITH MORE DERIVATIVES, YOU PICK UP OTHER DIMENSIONFUL PARAMETERS TO COMPENSATE.

↳ TYPICALLY GET $k/\Lambda$ SUPPRESSION

$A|f\rangle = |g\rangle \Rightarrow \int dy\, A(x,y)\, f(y) = g(x)$

$\sum_j A_{ij} f_j = g_i$

[UP TO SOME FACTORS OF $\Delta x$]

WHAT DOES THIS MEAN

Green's functions: $f = A^{-1} g$

## LINEAR DIFF. OPERATORS

$A(x,y)$ IS WRITTEN TO LOOK NONLOCAL
WE CAN ALSO WRITE IT IN AN "EXPANSION IN LOCALITY" IE AS A DIFFERENTIAL OPERATOR

$$A = \sum_n a_n(x) \left(\frac{d}{dx}\right)^n$$

DIFFERENTIAL OPERATOR

LOCAL FUNCTION

* REMARK: SEE SG § 3.2 FOR NORMAL FORM
↳ GIVES SCHRÖDINGER-TYPE EQ.

IN PHYSICS, TYPICALLY $\left[a_2(x)\frac{d^2}{dx^2} + a_1(x)\frac{d}{dx} + a_0(x)\right] f$

DEF: $\tilde{f} = h(x) f(x) \to$

$$(a_2 h) f'' + (a_1 h + 2 a_2 h') f' + (a_0 h + a_1 h'$$

SET $=0$; $h = \exp\left[-\frac{1}{2}\int_0^x \frac{a_2(y)}{a_1(y)}\, dx\right]$

then: GET EQ OF FORM

$\left[b_2(x)\left(\frac{d}{dx}\right)^2 + b_0(x)\right] f \longrightarrow$ SCHRÖ. EQ.

USEFUL TO DEF A GENERALIZED INNER PRODUCT:

$$\langle f | g \rangle_w = \int_D \underline{\underline{w(x)}} \, f^*(x) \, g(x) \, dx$$

FUNCTIONS OBEYING BOUNDARY CONDITIONS

WEIGHT FUNCTION
eg. IN SPHERICAL COORDS METRIC COMES w/ $r^2$

SO HAVING A WEIGHT HERE IS ACTUALLY RATHER NATURAL.

eg also in curved space WHERE $d^4x \to \sqrt{|g|}\, d^4x$

THEN WE ALSO WANT SENSE OF HERMITICITY

$$\langle f | A g \rangle_w = \langle A^+ f | g \rangle_w$$

LIN. DIFF. OPERATOR

WANT: $\int_D w(x) \, f^*(x) \left( \sum_n a_n(x) \left(\frac{d}{dx}\right)^n \right) g(x) \, dx$

MOVE OVER TO ACT ON $f(x)$

S.T. EXPR. STAYS SAME.

HOW TO DO IT? INTEGRATION BY PARTS.

USEFUL EXAMPLE: 1D MOMENTUM

$$\boxed{P = -i \frac{d}{dx}}$$

$\langle f | g \rangle_{w=1} = \int f^*(x)\, g(x)\, dx$ USUAL $L^2$ NORM

WAVEFUNCTIONS IN QM ($\mathbb{C}$ functions)
SQUARE INTEGRABLE → BC @ ∞

$\frac{d}{dx}(fg) = f'g + fg'$

$$\langle f | Pg \rangle = \int f^*(x) \left(-i \frac{d}{dx}\right) g(x)\, dx$$

$$= -(-i) \int \left(\frac{d}{dx} f^*(x)\right) g(x)\, dx - i \underbrace{\int \frac{d}{dx}(f^* g)\, dx}_{\text{TOTAL DERIV.}}$$

$$= \int \left(-i \frac{d}{dx} f(x)\right)^* g(x)\, dx + \int {}^{d}/_{dx}\, Q\, dx \qquad = \frac{d}{dx} Q$$

$$= \langle Pf | g \rangle + \underbrace{\int \frac{d}{dx} Q\, dx}_{= Q \,=\, 0}$$

$P^\dagger = P$
SELF ADJOINT (HERMITIAN)

as long as f & g SATISFY BC

BC IMPORTANT! PART OF DEFINITION OF HERMITIAN COND.

OBS: ROLE OF THE $i$! REQ. FOR HERMITICITY
→ $\mathbb{R}$ eigenvalues, ORTHOG EIGENFUN.

## STURM-LIOUVILLE PROBLEMS

IT IS COMMON IN PHYSICS TO HAVE FUNCTIONALS (like HAMILTONIANS) of THE FORM:

(SG 1.178)

$$S[f] = \int_{x_1}^{x_2} \left[ \frac{1}{2} p(x) (f'(x))^2 + \frac{1}{2} q(x) f(x)^2 \right] dx$$

$f(x) = 0$ @ $x_1, x_2$

& SOME NORMALIZ, eg. $\int_{x_1}^{x_2} f^2 \, dx = 1$

IN STONE & GOLDBART § 1.5 THIS IS TREATED AS A LAGRANGE MULTIPLIER PROBLEM.

I'D LIKE TO WRITE AS

$$S[f] = \int_D \frac{1}{2} f(x) \, \Theta \, f(x) \, dx$$

BECAUSE THEN EOM IS

$$\Theta f(x) = 0.$$

PROBLEM IS $\frac{1}{2} p(x) \left(\frac{d}{dx} f(x)\right) \left(\frac{d}{dx} f(x)\right)$

INT BY PARTS:

$$= -\frac{1}{2} f(x) \frac{d}{dx} \left[ p(x) \frac{d}{dx} f(x) \right]$$

END UP W/ $\boxed{-[p(x) f'(x)]' + qf = 0}$

sturm-liouville problem comes up over & over again.

RECALL: REDUCING GENERAL $2^{nd}$ O ~~LIN~~ LIN DIFF OP TO SCHRÖDINGER FORM,
WE CAN DO SOMETHING SIMILAR TO REDUCE TO STURM-LIOUVILLE FORM

SG. P.176

$$A = a_2\left(\frac{d}{dx}\right)^2 + a_1 \frac{d}{dx} + a_0$$

functions of $x$

TO AVOID SINGULAR POINTS → SUPPOSE $a_2 > 0$ over domain & $a_i \in \mathbb{R}$

DEF. $W = \frac{1}{a_2(x)} \int_{x_1}^{x} \frac{a_1(y)}{a_2(y)} dy$ WEIGHT!

POSITIVE ON $(x_1, x_2)$

$$Af = \frac{1}{w}(w a_2 y')' + a_0 y$$

$$\frac{1}{w}\left[\underbrace{w' a_2}_{w a_1 - a_2' w} y' + \cancel{w a_2' y'} + w a_2 y''\right]$$

NOW CHECK: $\langle f | A g\rangle \overset{?}{=} \langle Af | g\rangle$

$$\langle f | A g\rangle = \int dx\, w \frac{1}{w}(w a_2 g')' f^* + \cancel{a_0 f^* g}$$

$$-\left(\langle Af | g\rangle = \text{---}\text{''}\text{---}\ (w a_2 f^{*\prime})' g + \cancel{a_0 f^* g}\ \right]$$

$$= \int \frac{d}{dx}(w a_2 g' f^*) - \cancel{w a_2 g' f^{*\prime}}$$

$$- \int \frac{d}{dx}(w a_2 g f^{*\prime}) + \cancel{w a_2 g' f^{*\prime}}$$

$= 0$ for DIRICHLET or NEUMAN.

$$= \left[w a_2 g' f^* - w a_2 g f^{*\prime}\right]_{x_1}^{x_2}$$

$= Q$

EIGENVALUE PROBLEM B

$$\underbrace{Af}_{} = \lambda f$$

$$\frac{1}{w}(w a_2 f')' + a_0 f = \lambda f$$

$$\boxed{(w a_2 f')' + w a_0 f = \lambda w f}$$

so Stone & Goldbart offer the following hint:

IF YOU SEE AN EIGENVAL EQ. W/ WEIGHT ON RHS, SUSPECT IT IS SELF-ADJOINT W/RT $w$.

$\mathbb{R}$ eigenvals
ORTHOG EIGENF.

(PARTIAL) DIFFERENTIAL EQUATIONS

MANY INDEP VARS, eg SPACE + TIME

↑

not integrable in general!

BAG OF TRICKS APPROACH

1 INDEP. VAR. USUALLY TIME

SOLUTION EXISTS & IS UNIQUE (PICARD THM)

$y' = F(x, y)$

$$\begin{pmatrix} f_1' = f_2 \\ f_2' = f_3 \\ f_3' = f_4 \\ \vdots \\ f_n' = af_{n-1} \cdots \end{pmatrix}$$

TRICK TO WRITE $n^{th}$ Ⓞ ODE AS $1^{st}$ Ⓞ ODE IN HIGHER DIM

GRAPHICALLY: VECTOR FIELDS THAT HAVE TO BE INTEGRATED TO DETERMINE FLOWS.

↓

these flows ~~form~~ foliate manifolds — becomes a geometric question

conversely: easy to solve numerically

(P) DIFF EQ

NONLINEAR

contains powers of $f(x)$

or $[(d/dx)^k f(x)]$

$\downarrow$

cannot be written as $A f$

what to do if you're faced w/a nonlinear DIFF EQ?

GO HOME & RECONSIDER YOUR LIFE CHOICES

LINEAR : $A f(x)$

$= 0$

HOMOGENEOUS

just solve it

$= g(x)$ ← SOME OTHER FUNCTION

INHOMOGENEOUS

green's functions

↑

PIECE TOGETHER SOLUTION