TODAY: REV of LIN ALG
& INTRO TO FUNCTION SPACES AS LIN SP.

LECTURE 2: VECTOR SPACES — 26 SEPT

GOOD REFS: Matthews & Walker Math. Methods of Phy.
↳ CHAPTER 6, incl. nice exercise; rel to ∞ dim.

also ~~Goertzel~~ GOERTZEL & TRALLI
"Some Math. Methods of Physics"
Ch. 4, 8

You should already be familiar w/ the idea that QM = ($\mathbb{C}$) Linear Algebra.
→ WE WILL USE THIS AS A BASIS TO UNDERSTAND (LINEAR) DIFFERENTIAL EQUATIONS

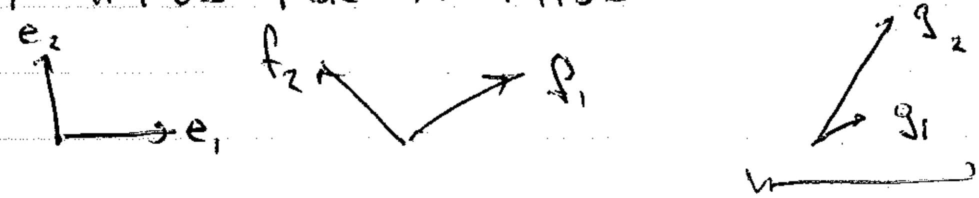
$\vec{V}$ → VECTOR SPACE $\xleftrightarrow{\text{in QM}}$ state space

VECTORS: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix} = \sum_i x_i \underline{e}_i = \sum_i x_i |e_i\rangle$

BASIS

just numbers

MANY CHOICES FOR A BASIS



$|x\rangle = \sum x_i |e_i\rangle = \sum x_i' |f_i\rangle$

BUT THE STATE IS BASIS INDEP

not a great choice!

~~TRIAL~~ SOME PROPERTIES $\rightarrow$ see, eg, APPENDIX of TEXTBOOK

VECTOR ADDITION: 2 VECTORS $\rightarrow$ VECTOR

IMPORTANT FOR SUPERPOSITION

$+ : V, V \rightarrow V$ COMMUTATIVE, ASSOCIATIVE

RESCALING: VECTOR & SCALAR $\rightarrow$ VECTOR

$\circ : V, \# \rightarrow V$ DISTRIBUTIVE

w/ IDENTITY & INVERSE ELEMENTS FOR BOTH OPS.

V^*: DUAL SPACE $\sim$ ROW VECTORS (UP TO CONJUG) KETS

DEFINED AS LINEAR MAPS THAT TAKE ELEMENTS OF A VECTOR SPACE TO #.

$\langle x | y \rangle$ IS A NUMBER

so $\langle x | : |y\rangle \rightarrow \#$

SIMILARLY: $|y\rangle : \langle x| \rightarrow \#$ (DUAL)

$$\begin{pmatrix} x_1^* & x_2^* & \cdots \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \end{pmatrix} = \sum x_i y_i$$

DUAL SPACE REALLY IS A DIFFERENT SPACE
$(\langle x| + |y\rangle)$ DOESN'T MEAN ANYTHING.

DUAL BASIS: $\underline{e}_i^*$ OR $\langle e_i| \neq |e_i\rangle$

USUALLY WE HAVE AN INNER PRODUCT TO RELATE V + V^*.

"DOT PRODUCT"

BUT USUALLY $\langle e_i|e_j\rangle = \delta_{ij}$

$\langle\cdot|\cdot\rangle : V \times V^* \longrightarrow \#$

such that $\langle x|y\rangle = \langle y|x\rangle^*$
and:

LINEAR IN V: $\langle x| ay_1 + by_2\rangle = a\langle x|y_1\rangle + b\langle x|y_2\rangle$
ANTILINEAR IN V^*: $\langle ax_1 + bx_2|y\rangle = a^*\langle x_1|y\rangle + b^*\langle x_2|y\rangle$

SO WE WRITE (IN QM) DAGGER:

$\langle x| \sim |x\rangle^\dagger$ in the sense that:

$$\dagger : |x\rangle \rightarrow \underbrace{\langle x, \cdot\rangle}$$

lin. funtion that takes kets to #.

the inner product of the basis vectors is the <u>metric</u>

$$\langle e_i | e_j \rangle = g_{ij}$$

SO NAMED BECAUSE IT GIVES US A WAY TO MEASURE ANGLES & DISTANCES.

eg $\| |x\rangle \|^2 = \langle x | x \rangle$

<u>heuristic</u> $\langle x | y \rangle = \text{"}\cos\theta\text{"} \sqrt{\langle x|x\rangle \langle y|y\rangle}$

OPERATORS / MATRICES / LINEAR TRANSFORMATIONS

$$A\underline{x} = \sum_j A_{ij} x_j \leftarrow \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & \dots \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} A_{11}x_1 + A_{12}x_2 \\ \vdots \end{pmatrix}$$

$$= \sum_{ij} A_{ij} |e_i\rangle\langle e_j| \sum_k x_k |e_k\rangle$$

$$= \sum_{ijk} A_{ij} x_k |e_i\rangle \langle e_j | e_k \rangle$$

RECALL $\mathbb{1} = \sum |e_i\rangle\langle e_i|$

METRIC

$$= \sum_{ij} A_{ij} x_j |e_i\rangle$$

EIGENVECTORS

$$A|x\rangle = \lambda |x\rangle$$

$\uparrow$ $\mathcal{O}$ OPERATOR OBSERVABLE

$\uparrow$ # $\to$ eigenvalue, observation

eg AN ELECTRON HAS 2 SPINS, REPRESENTED IN SPINOR SPACE AS $|\uparrow\rangle$ $|\downarrow\rangle$

S_z OPERATOR is $\sim |\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|$

SPIN OF $|\uparrow\rangle$ IS: $\langle\uparrow|S_z|\uparrow\rangle = +1.$

eg. WRITING $|\psi\rangle = \sum_i c_i |E_i\rangle$

energy eigenstates

TIME EVOLUTION $e^{iHt}|\psi\rangle = \sum_i c_i e^{iE_i t}|E_i\rangle$

USEFUL REVIEW

HERMITIAN CONJUGATE / ADJOINT : $A^\dagger$ ($\dagger = T*$)

s.t. $\langle y|Ax\rangle = \langle A^\dagger y|$

HERMITIAN MATRIX : $A^\dagger = A$

OBSERVABLES $\longrightarrow$ HAS $\mathbb{R}$ EIGENVALUES

GENERATES ~~___~~ UNITARY TRANSF.

$u^\dagger u = 1$

HERMITIAN MATRICES (OPERATORS)

- REAL EIGENVALUES

$$\langle Ax|x\rangle = \langle x|Ax\rangle$$
$$= \lambda^* \langle x|x\rangle \qquad = \lambda \langle x|x\rangle$$

- EIGENVECTORS ARE ORTHOGONAL

$$\langle Ax_i|x_j\rangle = \langle x_i|Ax_j\rangle$$
$$= \lambda_i \langle x_i|x_j\rangle \qquad = \lambda_j \langle x_i|x_j\rangle$$

but if $\lambda_i \neq \lambda_j$, then $\langle x_i|x_j\rangle \equiv 0$

NOW WE WANT TO CONNECT ALL OF THIS TO FUNCTION SPACES (∞ DIMENSIONAL!)

of course, this is what relates wave mechanics to matrix mechanics.

eg CONSIDER $[0,1]$ w/ FUNCTIONS f s.t. $f(0) = f(1) = 0$.

IDENTIFY: $|e_n\rangle = \sqrt{2}\,\sin(n\pi x)$

discrete basis of ∞ dim f'n space

$$f(x) = \sum_n c_n \sqrt{2} \sin(n\pi x)$$ BTW: $x = \ell / L$

$$c_n = \int_0^1 \sin(n\pi x)\, f(x)\, dx$$

$$|f\rangle = \sum_n c_n |e_n\rangle$$

$$c_n = \langle e_n | f \rangle$$

INNER PRODUCT

SOME THINGS YOU NOTICE:

FUNCTION SPACE HAS A <u>DOMAIN</u> → eg $[0,1]$
& <u>BOUNDARY COND.</u> → $f(0) = f(1) = 0$

need "sufficiently nice" functions; maybe C^∞ ...

WHAT COULD GO ~~GOES~~ WRONG IF WE USED DIFFERENT BOUNDARY COND?
→ is $\lambda_1 f_1(x) + \lambda_2 f_2(x)$ STILL IN THIS SPACE?

THE INNER PRODUCT IS AN INTEGRAL ← @ LEAST IN THIS CASE

$$\langle g | f \rangle = \int_D g^{(*)}(x)\, f(x)\, dx \quad \Big] \; L^2 \text{ NORM}$$

(*) for cx fun.

eg if we used e^{ikx}

SLIGHTLY MORE GENERALLY,

$$\langle g|f\rangle = \int_D w(x)\, g^*(x)\, f(x)\, dx$$

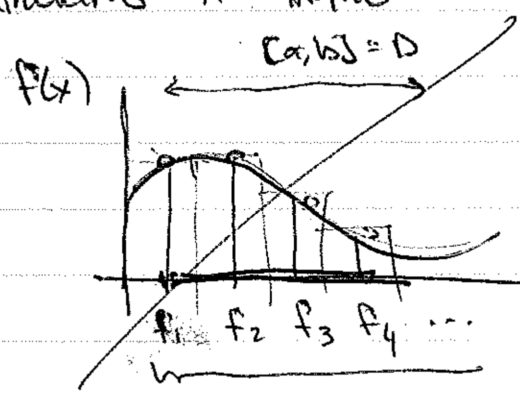
↑ WEIGHT FUNCTION

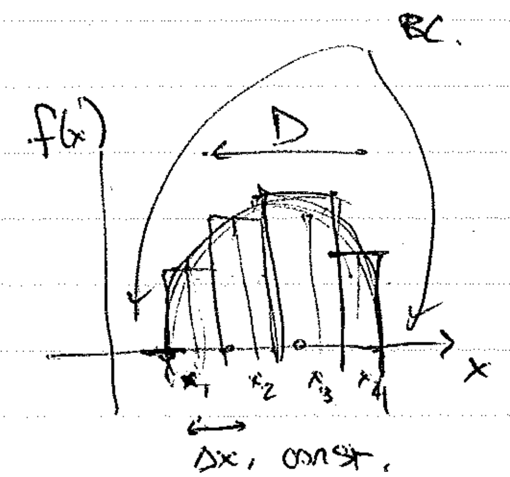
things I am not too worried about

↳ BUT THAT YOU SHOULD BE AWARE OF

- CONVERGENCE (cf BUTKOV p. 488)
- OTHER DEF. OF NORMS
- COMPLETENESS

HAMMERING IT HOME:





DISCRETIZED VERSION OF CONTINUOUS INTERVAL

THIS IS A 4D SPACE w/

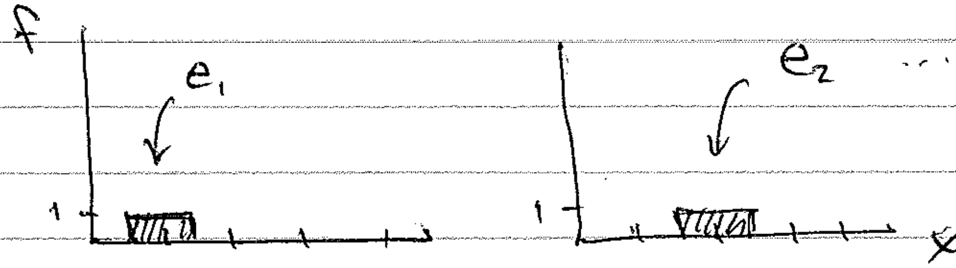
BASIS $e_1 = (1,0,0,0)$ ← VALUE of $f(x_1)$

$e_2 = (0,1,0,\ldots)$

$\vdots$

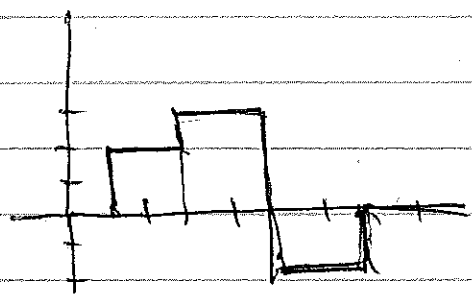
LINEAR VEC. SPACE, CLEARLY.

BASIS VECTORS:



if $\Delta x = 1$

<u>LIN. COMB</u>



$$|f\rangle = 2|e_1\rangle + 3|e_2\rangle - 2|e_3\rangle$$

in general: $|f\rangle = \sum_{i=1}^{4} c_i |e_i\rangle$

$$c_i = \langle e_i | f \rangle$$

note: Δx & dx ARE DIFFERENT

hybrid def

$$\int_{x_1 - \Delta x}^{x_4 + \Delta x} f(x) \left(\frac{e_i(x)}{\Delta x} \right) dx$$

to normalize

What happens as Δx gets small?

MATRIX
$$\begin{pmatrix} 1 & -1 & 0 & \cdots \\ \vdots & & & \\ & & & \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} f_1 - f_2 & \cdots \\ \vdots & \\ & \end{pmatrix}$$

I CAN IMAGINE SUCH AN OPERATOR ACTING ON OUR DISCRETIZED FUNCTION SPACE

($f_1 - f_2$ IS A DIFFERENCE BETWEEN TWO NEAR. NEIGHB. POINTS

SO:
$$\frac{1}{\Delta x}\begin{pmatrix} 1 & -1 & \\ & & \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} \frac{\Delta f(x_1+\Delta x)}{\Delta x} & \cdots \\ & \end{pmatrix}$$

so we see how the derivative pops out.

OBSERVATION: DIFFERENTIAL OPERATORS ARE "MATRICES" ACTING ON FUNCTION SPACE.

Q: only nearest neighbor... how to get nonlocal?

BUT THIS ALSO EXPLAINS WHY PHYSICS IS WRITTEN W/RT DIFF $\mathcal{O}$ OF LOW DIM
$\longrightarrow$ LOCALITY

IDENTITY OPERATOR

$$\mathbb{1} = \sum_i |e_i\rangle\langle e_i|$$

$$\mathbb{1}\,|f\rangle = \sum_i^4 |e_i\rangle\langle e_i| \sum_j f_j |e_j\rangle$$

$$= \sum_{i=1}^4 f_i |e_i\rangle \langle e_i|e_j\rangle$$

$$\frac{1}{\Delta x}\int_{x_i-\Delta x}^{x_i+\Delta x} e_i(x)\, e_j(x)\, dx \quad \left\{ \begin{array}{l} \delta_{ij} \text{ for FINITE DIM.} \\ \text{NICE METRIC} \end{array} \right.$$

GENERALIZATION:

$$f(x) = \int_D \delta(x-y)\, f(y)\, dy$$

$f(x)$ is the coeff of $|x\rangle$

ie this is the exp for a SINGLE c_i in $\sum c_i |e_i\rangle$

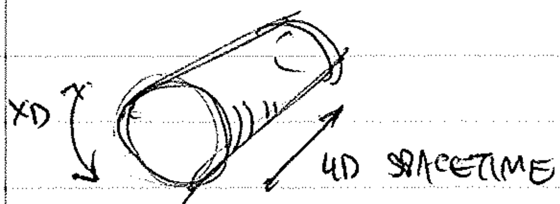
NEXT: this afternoon: DISC. SEC.

Wed: DIFF EQ. AS OP. ACTING ON THIS VECTOR SPACE

in particular: given

$$Ax = y \longrightarrow x \stackrel{?}{=} A^{-1}y$$

ASIDE: HOW I'VE USED THIS

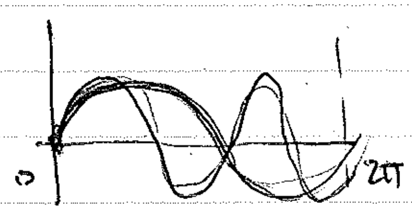


WAVEFUNCTIONS (really: FIELDS) MUST BE CONTINUOUS AROUND WHOLE SPACE

→ BC. OVER COMPACT XD.

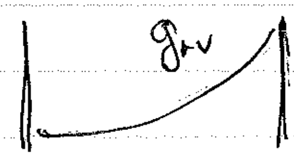
WE CARE ABOUT STATES OF WELL DEF. ENERGY

→ KK TOWER ⟷ OCTAVES



etc.

eg sines. → just Fourier series.

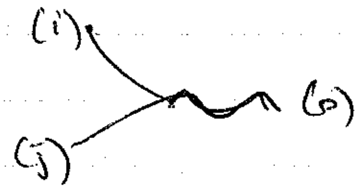


MAKE IT MORE COMPLICATED.

WARPED XD ⇒ WAVEFUNCTIONS ARE REDSHIFTED

↳ get more complicated basis of fns

BUT STILL ORTHOGONAL →



overlap integral ~ δ_{ij}