HOMEWORK 5: **Killing Fields, Black Holes**

COURSE: Physics 208, General Relativity (Winter 2017)
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DUE DATE: Tuesday, Feb 21 in class... or, you know, like... whenever.

You are required to complete the Reading Assignment and Essential Problems below. Please let me know if these are too time intensive. You are invited to explore the 'extra' problems as they apply to your goals for this course: Mathematical Problems develop geometric intuition, while Phenomenological Problems are applications of relativity.

Reading Assignment

Read the following topics. You may choose to read the analogous topics in an appropriate textbook or reference of your preference. Most of this reading is meant to be complementary to the approach in the lectures.

- Finish chapter 12 of Hartle
- We will not spend much time on more realistic black holes, but feel free to peruse chapters 13 – 15 as you are interested. For an excellent set of lecture notes, see `gr-qc/9707012`.
- You can start reading chapter 18 on cosmological models—we will not discuss them too much in class, but we will play with FRW metrics in upcoming homework.
- The Einstein equation, which we are rapidly approaching, is described in chapter 21.

The Schwarzchild metric is:

$$ds^2 = \left(1 - \frac{r_s}{r}\right) dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 - r^2 d\Omega^2 \ . \tag{0.1}$$

We've written $r_s = 2GM$ as the Schwarzschild radius.

CAVEAT EMPTOR: as always, do what I mean, not necessarily what I say.

Essential Problems

1 Schwarzschild Coordinate Time as $r \to r_s$

Consider a test particle dropped at radius r_0 away from a Schwarzschild black hole. The particle has a conserved energy that is chosen so that it has zero initial velocity at infinity. We showed in class that the velocity in the test particle's frame is

$$\frac{dr}{d\tau} = \sqrt{\frac{r_s}{r}} \ . \tag{1.1}$$

This is perfectly well behaved as r passes through the Schwarzschild radius, r_s. In this problem, we check what an observer[1] Use the energy conservation condition to show that

$$\frac{dt}{dr} = \frac{dt/d\tau}{dr/d\tau} = -\frac{\sqrt{r/r_s}}{1 - r_s/r} \ . \tag{1.2}$$

Integrate this to obtain:

$$t - t_0 = -\frac{2}{3\sqrt{r_s}}\left(r^{3/2} - r_0^{3/2} + 3r_s\sqrt{r} - r_s\sqrt{r_0}\right) + 2m\ln\frac{\left(\sqrt{r}+\sqrt{r_s}\right)\left(\sqrt{r_0}-\sqrt{r_s}\right)}{\left(\sqrt{r_0}+\sqrt{r_s}\right)\left(\sqrt{r}-\sqrt{r_s}\right)} \ . \tag{1.3}$$

Okay, you don't actually have to integrate that by hand. Use *Mathematica*. Or just check the result by integrating. Don't squander your youth. However, *do* plot $t(r)$ and $\tau(r)$ on the same plane to explicitly show that one is perfectly well behaved and the other gets cranky as $r \to r_s$.

Recall that the solution for $\tau(r)$ was

$$\tau - \tau_0 = \frac{2}{3\sqrt{s}}\left(r_0^{3/2} - r^{3/2}\right) \ . \tag{1.4}$$

2 Finishing your thesis when you're doomed

You accidentally fall into the event horizon of a black hole. Perhaps experimental observations of black hole event horizons are part of your dissertation. You want to finish writing your thesis before you reach the singularity. You have reasonable booster rockets on your spaceship. What is your best strategy to make sure you *maximize your time* to finish writing your thesis[2]?

3 All of those coordinate transformations

In class we went through a series of coordinate transformations to understand the nature of the event horizon of a Schwarzschild black hole. Please go through the exercise of re-deriving the following quantities. Note: you can leave r as an implicit function of each coordinate system so that, for example, you can just write $-r^2 d\Omega^2$.

1. In **tortoise coordinates**, we introduced

$$\widetilde{r} = r + 2r_s \log\left(\frac{r}{r_s} - 1\right) \ , \tag{3.1}$$

where 'log' means natural log. Derive the metric in $(t, \widetilde{r})$ coordinates and argue that this form means that light cones do not get 'smushed' as $r \to r_s$.

[1] The observer is stationary in the Schwarzschild coordinates. For example, it is at some finite r_{obs} and in an accelerated frame or otherwise sufficiently far away from the black hole that the acceleration from the black hole is negligible.

[2] Sure, nobody outside the event horizon will be able to read it and eventually you'll die... but this is how most people feel about their thesis, even in Minkowski space.

2. Write the metric in **Eddington–Finkelstein** coordinates, (u, r), where

$$v = t + \widetilde{r} \; . \tag{3.2}$$

What are the slopes of the lightcones, dv/dr in these coordinates? Thus argue that these coordinates describe a spacetime where things from $r > r_s$ can fall into $r < r_s$.

3. As we tried to derive a maximally extended description of Schwarzschild spacetime, we proposed using u and v coordinates:

$$v = t + \widetilde{r} \qquad\qquad u = t - \widetilde{r} \; . \tag{3.3}$$

The coordinate r is implicitly defined from $(v - u)$; argue that $r = r_s$ is pushed to infinity in these coordinates. Note that the 'infinity' is a logarithmic divergence. Write the metric in these coordinates.

4. To pull the event horizon back from infinity, we proposed exponentiated coordinates:

$$\widetilde{v} = e^{v/2r_s} \qquad\qquad \widetilde{u} = -e^{-u/2r_s} \; . \tag{3.4}$$

Write $\widetilde{v}$ and $\widetilde{u}$ in terms of the original (r, t) coordinates. Write the metric in $(\widetilde{u}, \widetilde{t})$ coordinates.

5. Finally, we moved to **Kruskal** coordinates which had a timelike and spacelike direction (rather than two null or 'lightcone' directions),

$$T = \frac{\widetilde{v} + \widetilde{u}}{2} \qquad\qquad R = \frac{\widetilde{v} - \widetilde{u}}{2} \; . \tag{3.5}$$

Write these in terms of the original (r, t) coordinates. Write the metric in (R, T) coordinates.

6. Draw the maximally extended Schwarzschild spacetime as given by Kruskal coordinates. Draw the event horizon. Draw lines of constant t. Draw lines of constant r, including the singularity $r = 0$. Identify the four regions as 'inside the black hole', 'inside a white hole', and two causally disconnected spacetime regions.

Phenomenological Problems

4 Killing vectors of 3D

Hartle Example 8.6 & Problem 8.8. We motivated Killing vectors as directions that leave the metric unchanged (isometries). We argued that in D dimensions, there are at most $\frac{1}{2}d(d+1)$ Killing vectors. Consider 3D flat space, $ds^2 = dx^2 + dy^2 + dz^2$. The three 'obvious' Killing vectors are in the three Cartesian directions since the metric is independent of these coordinates. Shift to spherical coordinates: identify another Killing vector. Based on this, what are the two remaining Killing vectors? (You may answer either explicitly or qualitatively.)

5 Gravitational Redshift

Consider the gravitational redshift of a photon in a Schwarzschild spacetime (HARTLE SECTION 9.2). The time direction is a good Killing vector, $K_{(t)}$ so that $K_{(t)} \cdot p$ is a conserved quantity.

An observer with 4-velocity u observes the photon with energy $E = p \cdot u$. This is what the observer identifies with $\hbar\omega$. Recall that $u^2 = 1$ (why?). Using the Schwarzschild metric, show that

$$u^\mu = \left(1 - \frac{rs}{r}\right)^{-1/2} K^\mu_{(t)} \ . \tag{5.1}$$

What is the ratio of photon frequencies ω measured by an observer at some position $r = R$ versus $r = \infty$ (where ∞ means 'very far away where space is effectively Minkowski')?

Compare this to what we calculated in class using the principle of equivalence in the background of a Newtonian source:

$$\omega_\infty = \left(1 - \frac{GM}{R}\right) \omega_R \ . \tag{5.2}$$

Mathematical Problems

6 Killing Vectors of Spherical Symmetry

Consider the Killing vectors in Problem 4. Show that the three rotations may be written in spherical coordinates as

$$R = \partial_\phi \qquad S = \cos\phi\, \partial_\theta - \cot\theta \sin\phi\, \partial_\phi \qquad T = -\sin\phi\, \partial_\theta - \cot\theta\cos\phi\, \partial_\phi \ . \tag{6.1}$$

Calculate the commutators (Lie derivatives) of these fields. Do these look familiar from quantum mechanics? This is the algebra[3] of SU(2).

7 Building a Maximally Symmetric Space

One trick to construct a maximally symmetric d-dimensional space is to embed it into a flat $(d+1)$-dimensional space. Consider, once again, flat 3D space. This is maximally symmetric, you identified the $\frac{1}{2}d(d+1)$ Killing vectors in problems 4 and 6. From this we can construct a maximally symmetric, curved 2D space by imposing a constraint

$$x^2 + y^2 + z^2 = R^2 \ . \tag{7.1}$$

[3]**Lie groups** are groups that are also manifolds. The **Lie algebra** of the group is precisely the set of Lie derivatives at the origin. Often representation theory of continuous groups is associated more closely with discrete group theory, but in many ways it is naturally understood as a funny manifold.

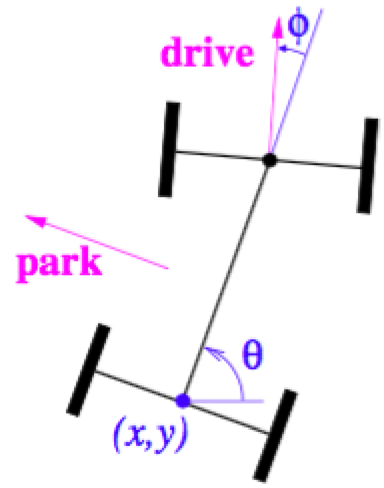

Figure 1: Fig. 11.5 from Stone & Goldbart. Coordinates for car parking.

By imposing this constraint we have broken many of isometries of the 3D space. Which isometries are broken? (Which Killing vectors are not Killing vectors of the constrained 2D space?) How many Killing vectors are left over? Based on the number of Killing vectors and the dimensionality of the space, argue that the sphere is maximally symmetric.

Remark: de Sitter and anti-de Sitter space are maximally symmetric 4D spacetimes that are formed analogously by considering 5D Minkowski space and projecting onto hyperboloid-y submanifolds.

8 Lie Derivatives and the Parallel Parking Problem

This question was originally on the Physics 231 homework set (Fall 2016, HW7).

Southern California was the birthplace of car culture in the United States[4]. One of the consequences is that Southern Californians are notoriously bad at parallel parking—*fit my car in there? That's impossible!* The point of this problem is to mathematically prove that parallel parking into a space that is strictly greater than the length of your car is possible.

This problem is essentially exercise 11.1 in Stone and Goldbart, the discussion in the book may be useful. Recall from lecture that parking a car is an *anholonomic* system. The configuration space of a car is four-dimensional, given by the position of the car (say the center of mass, or the center of the rear axle), the angle with respect to some reference axis (say, the sidewalk), and the angle of the front wheels used for steering. Call these (x, y, θ, ϕ), as shown in the figure.

[4] Uh... citation needed. What do I look like, a historian?

A convenient basis of vector fields are:

$$\mathbf{drive} = \cos\phi\left(\cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}\right) + \sin\phi\frac{\partial}{\partial\theta} \tag{8.1}$$

$$\mathbf{steer} = \frac{\partial}{\partial\phi} \tag{8.2}$$

$$\mathbf{skid} = -\sin\phi\left(\cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}\right) + \cos\phi\frac{\partial}{\partial_\theta} \tag{8.3}$$

$$\mathbf{park} = -\sin\theta\frac{\partial}{\partial x} + \cos\theta\frac{\partial}{\partial y}\ . \tag{8.4}$$

Calculate the Lie brackets of all combinations of these vector fields. There are six to calculate.

A driver can only use $\pm$**drive** and $\pm$**steer** to maneuver the car. Use the geometric interpretation of the Lie bracket (commutator) to explain how a suitable sequence of motions (forward, reverse, and turning the steering wheel) can be used to move the car *sideways* into a parking space.

Advice from Burke, *Applied Differential Geometry* (1985, p. 133):

> Just remember to steer, drive, steer back, drive some more, steer, drive back, steer back, drive back: $SDS^{-1}DSD^{-1}S^{-1}D^{-1}$. You have to reverse steer in the middle of the parking place. This is not intuitive, and no doubt is part of the problem with parallel parking. Thus from only two controls you can form the vector fields **steer**, **drive**, **rotate**, and **slide**[5], which span the full tangent space at every point. You can move anywhere in the four-dimensional configuration space.

If you need more help, you can try looking for discussions on the web. Here are a couple:

- https://www.math.wisc.edu/~robbin/parking_a_car.pdf
- https://rigtriv.wordpress.com/2007/10/01/parallel-parking/ .

[5] Burke uses a slightly different set of vector fields than Stone & Goldbart.