Homework 4: **Schwarzschild Metric**

Course: Physics 208, General Relativity (Winter 2017)
Instructor: Flip Tanedo (flip.tanedo@ucr.edu)
Due Date: Tuesday, Feb 14 in class... or, you know, like... whenever.

You are required to complete the Reading Assignment and Essential Problems below. Please let me know if these are too time intensive. You are invited to explore the 'extra' problems as they apply to your goals for this course: Mathematical Problems develop geometric intuition, while Phenomenological Problems are applications of relativity.

Because this problem set is late (and the previous one was long), this will be a short set.

Reading Assignment

Read the following topics. You may choose to read the analogous topics in an appropriate textbook or reference of your preference. Most of this reading is meant to be complementary to the approach in the lectures.

- Make sure you're familiar with chapters 20 and 21 of Hartle (you can skip the linear equations)
- Finish chapter 9 continue to chapter 12 of Hartle
- Read chapter 11.1 of Hartle if you're interested in lensing.

Reminder: The Schwarzschild metric is:

$$ds^2 = \left(1 - \frac{r_s}{r}\right) dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 - r^2 d\Omega^2 \ , \tag{0.1}$$

where $r_s = 2GM$ and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$.

Essential Problems

1 Gravitational Index of Refraction

For a spherically symmetric metric, the speed of light according to a local observer (at the origin) is measured by looking at the [small] displacement dr divided by some [small] proper time $d\tau$. Indeed, the observer measures $dr/d\tau = 1$.

Because of gravitational time dilation, this is no longer true for some observer located a finite distance away from the measurement. In the spherically symmetric metrics that we've encountered, the time coordinate t is that given by a clock that's far from the gravitational source. In other words, the speed of light observed a distance r away is given by dr/dt. What is the speed of light observed at distance r in the Schwarzschild metric? What is the speed of light in the 'Newtonian' metric, $ds^2 = g_{00}(r)dt^2 - d\mathbf{x}^2$, where $g_{00}(r) = (1 - r_s/r)$? What is the corresponding [position-dependent] **index of refraction** for these two spaces?

If we used the Newtonian metric to measure gravitational lensing (See Hartle chapter 11.1), how would our predictions compare to the 'correct' lensing angle predicted from the Schwarzschild metric?

2 Christoffels of Schwarzschild

Use the *Mathematica* package on Jim Hartle's webpage[1] to calculate the Christoffel symbols of the Schwarzschild metric and the four geodesic equations.

Please calculate the Γ^t_{rt} coefficient by hand and compare to the *Mathematica* results.

Phenomenological Problems

3 Precession of the Perhelion of Mercury

In class we derived a 'constant energy equation' for the Schwarzschild metric. We identified the correction to the ordinary Newtonian central force potential. Treating this correction as a perturbation to the Newtonian potential and using the results from ordinary mechanics, calculate that the precession of the perhelion of Mercury is indeed 43 arc-seconds per century. It is useful to use:

$$r(\phi) = \frac{\alpha}{1 + e\cos\left((1-\varepsilon)\phi\right)} \qquad \alpha \equiv \frac{\ell^2}{GMm^2} = \frac{2\ell^2}{r_s m^2} \ . \tag{3.1}$$

Here $e = 0.206$ is the eccentricity of the orbit of Mercury. You'll need to show that

$$\varepsilon = \frac{3r_s}{2\alpha} \ . \tag{3.2}$$

Use $r_s = 2.95$ km and $r_{\text{min}} = 4.6 \times 10^7$. Play with the results in Hartle's *Mathematica* notebook for Schwarzschild orbits.

Mathematical Problems

4 Birkhoff's Theorem

There's a fantastic result called **Birkhoff's Theorem** which says that spherically symmetric spacetimes must be time-independent and asymptotically flat. The first statement says that the metric is static, the second says that far away from the origin it looks like Minkowski space. Follow **Hartle, problem 21-18** to prove this theorem.

NOTE: Okay, so you use Einstein's equation for this... you have all of the ingredients to use Einstein's equation, but we haven't brought it up in class yet.

[1] http://web.physics.ucsb.edu/~gravitybook/mathematica.html