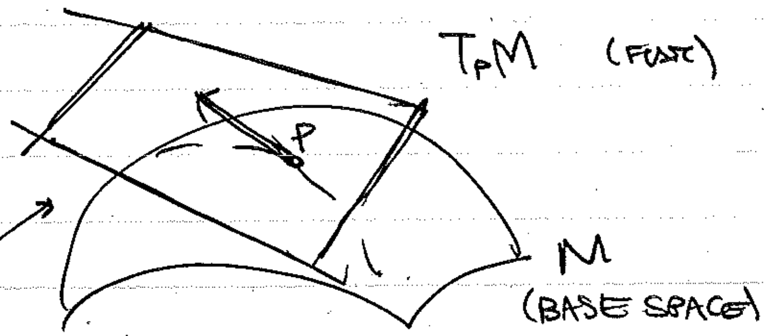
LEC 18: DIFF FORMS II 16 MARCH

REVIEW

mathematician's view of EINSTEIN'S EQUIV. PRINCIPLE:



flat vector space
LINEAR ALGEBRA LIVES HERE

METRIC: vectors $\longleftrightarrow$ dual vectors (1-forms)

@ this stage: totally symmetric

METRIC: DEFINED AS A FUNCTION ON THE BASE SPACE M, BUT ITSELF IS A LINEAR MAP (sym 2-lower-index-tensor) ON THE TANGENT SPACE @ EACH POINT.

So:

$$\begin{tikzcd} \text{1-vectors} \arrow[r, bend left, "\text{lin. func of}"] & \text{1-forms} \arrow[l, bend left, "\text{lin func. of}"] \end{tikzcd}$$

<u>ADDED STRUCTURE</u> : <u>MULTI</u>LINEAR MAPS ARE JUST TENSORS

CAN CALL $T^{\mu\nu}$ A "2-VECTOR"

$\eta_{\mu\alpha}\eta_{\nu\alpha}T^{\alpha\beta}$ IS A 2-LOWER-INDEX OBJECT.

BUT <u>NOT</u> WHAT WE CALL A "2-FORM"

$$T_{\cdot\cdot} = (\text{symmetric})_{\cdot\cdot} \oplus \boxed{(\text{antisym})_{\cdot\cdot}}$$

(trace) ⊕ (trace-less)

≡ 2-form

BY <u>DEFINITION</u>

.. odd def @ this point, but will be useful.

<u>K-form</u>: K-linear ~~map~~, totally antisym map from $(T_pM)^k \longrightarrow \mathbb{R}$

$$\omega^{(k)}(V_{(1)}, \cdots, V_{(k)}) = \#$$

$$= -\omega^{(k)}(V_{(2)}, V_{(1)}, \cdots V_{(k)})$$

etc.

s.t. in indices:

$$\omega_{\mu\nu\cdots} = -\omega_{\nu\mu\cdots}$$

for ordinary vectors, could slap them together to make weird tensors: $V^\mu W^\nu$ is a tensor.

TO MAKE "BIGGER" FORMS, NEEDED AN ANTISYMMETRIC WAY OF SLAPPING THINGS <u>TOGETHER</u>

$$\omega^{(k)} \wedge \rho^{(l)} = (-)^{kl} \rho^{(l)} \wedge \omega^{(k)}$$

k-form l-form

manifestly antisym (DEF. PRINCIPLE)

sum over indep.

CAN WRITE

$$\omega^{(k)} = \sum a_{i_1 \cdots i_k} \; \underbrace{\tilde{e}_{i_1} \wedge \cdots \wedge \tilde{e}_{i_k}}_{\text{basis for k-forms}}$$

often def w/ some factor of $1/k!$ or something ... we'll ignore

looks like a $k \times k$ determinant

eg. $\tilde{e}_1 \wedge \cdots \wedge \tilde{e}_k$ in $\mathbb{R}^n$ $\quad n > k$

takes k vectors, $V_{(1)}, \ldots, V_{(k)}$

& gives

$$\begin{array}{ccc} V_{(1)}^1 & \cdots & V_{(k)}^1 \\ \vdots & & \\ V_{(1)}^k & \cdots & V_{(k)}^k \\ \hline \vdots & & \\ V_{(1)}^n & \cdots & V_{(k)}^n \end{array}$$

this det.

$\tilde{e}_2 \wedge \cdots \wedge \tilde{e}_{k+1}$ GIVES DET of "ONE BOX BELOW"

etc.

$\tilde{e}_1 \wedge \tilde{e}_2 \wedge \cdots$ is redundant w/ $\tilde{e}_2 \wedge \tilde{e}_1 \wedge \cdots$ etc.

(HENCE THE $1/k!$)

<u>then</u>: calculus. these basis k-forms are identified with

$$\boxed{dx^{i_1} \wedge \dots \wedge dx^{i_k}}$$

dx^i <u>is</u> a basis 1-form defined by $dx^i(e_j) \equiv \delta^i{}_j$

∂_j

alternatively: <u>differential op</u> acting on coordinate function

"d of x"

cf $$df = (\partial_i f)\, dx^i$$
$$d(x^j) = (\partial_i x^j)\, dx^i$$
$$= \delta^j{}_i \, dx^i$$

d: exterior derivative

CAN GENERALIZE DIFFERENTIATION TO k-FORMS

d: k-form $\to$ (k+1)-form

$$d\omega = \sum_{i\ldots} (\partial_j \omega)\, dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

& we found, eg. for a 1-form A in $\mathbb{R}^3$

$$dA = (\nabla \times A)^i \, \underline{\varepsilon_{ijk}\, dx^j \wedge dx^k}$$

curl is secretly a 2-form!

recover one-index object when we define HODGE STAR

$$* \, dx^i \wedge dx^j = \frac{1}{2} g^{ia} g^{jb} \varepsilon_{abk}\, dx^k \quad \text{in 3D}$$

in n DIM, $\varepsilon_{i_1 \cdots i_n}$ is a valid tensor

<u>in gen:</u> $*$ turns k-form into (n-k)-form (w/ an ε tensor)

nb no surprise that k-form & (n-k) form carry same amt of info!

PROPERTIES OF THIS STRUCTURE $\Rightarrow$ GAUGE THY

<u>observe</u>: $d^2 = 0$

$$dd\omega \sim \Sigma\,(\partial_k \partial_j\, \omega)\, dx^k \wedge dx^j \wedge \dots$$

$= 0$

"d is nilpotent"

so if you have a <u>POTENTIAL</u>, eg
1-form potential A s.t. PHYSICS LIVES
IN $dA \equiv F$

then you can shift A by d(0-form)
& leave physics unchanged!

$$A \to A + d\alpha \;\Rightarrow\; F \to dA + d^2\alpha = dA = F \checkmark$$

so physics is defined up to a 0-form's
worth of data

$\uparrow$

GAUGE FIXING
(if you don't, you wreck havoc
on your path integrals)

this idea that dA contains physics but $d^2A \equiv 0$ is called <u>COHOMOLOGY</u> by the cognoscenti ... i mention it here for future poetry..

APPLIED TO $F = dA$, this gave $\boxed{dF = 0}$
½ of MAXWELL

DIFFERENTIAL FORMS WERE BORN TO BE INTEGRATED

dx^i - infinitesimal line element
stick a bunch together

$\int_\ell dx$ gives arclength ($\int f\, dx$ gives f summed along path)

$dx^i \wedge dx^j$ - infinitesimal area
analogous to $\underline{a} \times \underline{b}$

$\int dx^i \wedge dx^j$ gives 2D area

$\int_A f \, dx^i \wedge dx^j$ gives f SUMMED ALONG AREA

& so forth: for an n-dimensional space, the n-volume is given by integrating the volume form

$$\Omega = dx^1 \wedge \cdots \wedge dx^n$$

$\int_V \Omega$ = VOLUME. $\int f \Omega$ = f SUMMED OVER VOL

CALCULUS – in the most confusing way : "integrals of derivatives"

FUNDAMENTAL THM : $\underset{\substack{\uparrow \\ \text{interval } (a,b)}}{\int_I} \underbrace{f\,dx}_{df} = f(a) - f(b) = $ "$\int_{\partial I} f$"

GREEN'S THM : $\int_A (\vec{\nabla} \times \vec{V}) \cdot d\vec{A} = \oint_{\partial A} V \cdot dx$

DIV STOKES' THM : $\int_V \vec{\nabla} \cdot \vec{E}\, d(\text{vol}) = \oint_{\partial V} \vec{E} \cdot d\vec{A}$

are there more for 4D? 5D? ...

POETIC ABUSE OF NOTATION:

∂(space) = "boundary of space"

$\partial^2 = ?$ (HOMOLOGY)

dim $V = k+1$ (k+1) form k-form BOUNDARY OF V (k dim)

GENERAL: $\int_V d\omega = \int_{\partial V} \omega$

the integral of the (k+1) - form $d\omega$, ie the int of [ext. derivative of k-form ω],

is equal to the lower - dimensional integral of the k - form ω on the boundary of V, ∂V.

FUND THM OF CALCULUS — trivial interp of the above principle.

GREEN'S THM: $\omega = A_i\, dx^i$

1-form that contains same data as $\vec{A}$ (sometimes say: $\omega = \vec{A}^{\#}$, $\vec{A} = \omega^{b}$)

then: $\int d\omega = \int_{AREA} d(A_i dx^i)$

$\partial_j A_i\, dx^j \wedge dx^i$

$= \int_{AREA} (\underline{\nabla} \times \underline{A})\, d(\text{area})$

$\int_{\partial AREA} \omega = \oint A_i\, dx^i$ ← LINE INTEGRAL AROUND BNDY! (oriented)

DIVERGENCE THM (i'll be a little slick, to write it out rigorously gets a little tedious)

$$\omega = E_x\, dy\wedge dz + E_y\, dz\wedge dx + E_z\, dx\wedge dy$$

carries same data as $\vec{E}$
(related by $*$ ... but that just adds more notation)

$$d\omega = \underbrace{(\partial_x E_x + \partial_y E_y + \partial_z E_z)}_{\nabla\cdot\vec{E}}\ \underbrace{dx\wedge dy\wedge dz}_{\Omega}$$

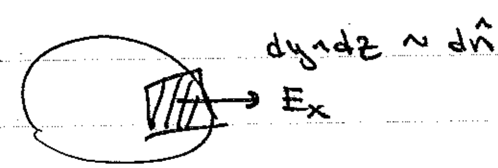
eg. b/c $\partial_y E_x\, dy\wedge dy\wedge dz \equiv 0$

$$\int_V d\omega = \int_V (\nabla\cdot\vec{E})\, dVol$$

$$\parallel$$

$$\int_{\partial V}\omega = \int_{\partial V} E_x\, dy\wedge dz + \cdots$$

$$= \int_{\partial V}\vec{E}\cdot d\vec{A}$$



OTHER RESULTS FROM VECTOR CALCULUS : $d^2 = 0$

CASE: ω is 0-form $[\omega = f(x)]$

$d\omega = df$, 1-form
$= \partial_x f \, dx + \cdots = \underline{\nabla} f \cdot d\vec{x}$

curl of grad

$d^2\omega = 0$
$= \underline{\nabla} \times \underline{\nabla} f \longrightarrow \boxed{\underline{\nabla} \times \underline{\nabla} = 0}$

since we noticed d(1-form) in 3D gives curl.

not nec. $\partial_x g$

CASE: ω is 1-form $[\omega = f_x \, dx + \cdots]$

$d\omega = (\vec{\nabla} \times \vec{f})_z \, dx \wedge dy + \cdots$

div of curl

$d^2\omega = 0$
$= \vec{\nabla} \cdot \vec{\nabla} \times f \Rightarrow \boxed{\vec{\nabla} \cdot \vec{\nabla} \times = 0}$

since we noticed that d(2-form) gives z of components that are the divergence

<u>OBSERVE</u>: $d^2\omega = 0$

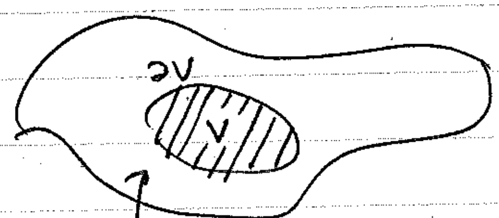
$$\Downarrow$$

$$\int_V d^2\omega = 0$$

$$\parallel$$

$$\int_{\partial V} d\omega = \int_{\partial^2 V} \omega$$

$$\partial^2 = (\text{boundary of})^2 \equiv 0$$



BUT ∂V IS <u>NECESSARILY</u> BOUNDARY-FREE

this is called <u>HOMOLOGY</u>

(vs $d^2 = 0 \longleftrightarrow$ COHOMOLOGY)

<u>nb</u>: this is just the tip of the iceberg of how diff. geometry connects <u>CALCULUS</u> to <u>TOPOLOGY</u>.

favorite manifold : SPACETIME
favorite integral : ACTION $\rightarrow S = \int \mathcal{L}$

evidently, LAGRANGIAN DENSITY IS AN n-form ON n-dim SPACETIME

HOW TO WRITE ACTION FOR ELECTRODYNAMICS?

E&M lives in $F = dA$, 2 form

need to "fill this out" into a 4-form
$dF \equiv 0$, so CAN'T USE d.

<u>CONCATENATE</u> : $F \wedge F$ OR $F \wedge * F$

WORKS IN 4D — WORKS IN ANY DIM

<u>which one</u>?

~~No: $F \wedge$~~

$$F \wedge \star F = \varepsilon_{\alpha\beta\gamma\delta} \underbrace{g^{\mu\alpha} g^{\nu\beta}} F_{\mu\nu} F_{\rho\sigma} \; dx^\rho \wedge dx^\sigma \wedge dx^\gamma \wedge dx^\sigma$$

DEPENDS ON METRIC!
→ CONTRIBUTES TO
STRESS ENERGY
$T_{\mu\nu} \sim \delta S / \delta g^{\mu\nu}$

WHAT DOES IT LOOK LIKE?
PICK $\mu, \nu = 0, 1$

$\Rightarrow g^{\mu\alpha} g^{\nu\beta} \Rightarrow \alpha\beta = 01$
$\Rightarrow \varepsilon_{\alpha\beta\gamma\delta} \Rightarrow \gamma\delta = 23$
$\Rightarrow dx \wedge \cdots \wedge dx \Rightarrow \rho\sigma = 01$

$F_{\mu\nu} F_{\rho\sigma} \sim E_x^2$

& indeed, we end up w/ $(\vec{E}^2 - \vec{B}^2)$ ✓

WHAT ABOUT $F \wedge F$?

$= F_{\alpha\beta} F_{\rho\sigma} \; dx^\alpha \wedge dx^\beta \wedge dx^\rho \wedge dx^\sigma$

pick $\alpha\beta = 0\,1$
$dx\wedge .. \Rightarrow \rho\sigma = 2\,3$

so $F_{..} F_{..} \sim \underline{E} \cdot \underline{B}$

IN TENSOR NOTATION

$\partial_\alpha A_\beta \; \cancel{\ldots} \; \partial_\delta A_\gamma$

implicit: antisym in $\alpha \leftrightarrow \beta$ & in $\delta \leftrightarrow \gamma$

integ. by parts:

$$= \partial_\alpha (A_\beta \partial_\delta A_\gamma) - A_\beta \underbrace{\partial_\alpha \partial_\delta}_{} A_\gamma$$

sym. → VANISHES ON $dx \wedge \ldots \wedge dx$

<u>total derivative</u>

$= 0$ for EM

... but not nec. for <u>NONABELIAN</u> gauge thy.

no coupling to $g^{\mu\nu}$... "no energy" → in gen: topological term

SYMMETRY APPROACH TO E&M

WRITE OUT ALL THE CANDIDATE 4-FORMS

↳ $F \wedge {*F} \longrightarrow \mathcal{L}_{EM}$

$F \wedge F \longrightarrow$ total deriv. (SURFACE TERM)

BUT ALSO... $\underbrace{*(F \wedge F)}_{\text{0-form}} \; F \wedge {*F}$

this is a 4-form

... BUT DIMENSIONAL ANALYSIS ($c = \hbar = 1$) GIVES A PREFACTOR THAT GOES LIKE $1/\Lambda^4$

↑

presumably some heavy UV scale ... irrelevant for low energy physics.

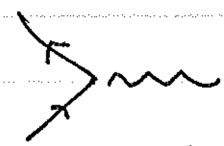
& so forth for more complicated terms.

SO WE FIND THAT $\mathcal{L}_{EM}$ is "UNIQUE" TO GOOD APPROX!

COUPLING TO MATTER

WANT SOMETHING LIKE $j^{\mu} A_{\mu}$

EM CURRENT POTENTIAL



CANDIDATE : $A \wedge \underbrace{*j}_{\text{3 FORM}}$

$$S = \int F \wedge *F + A \wedge *j$$

let's do this glibly:
INT BY PARTS:

$$S = \int \cancel{d(A \wedge *F)} - A \wedge d*F + A \wedge *j$$

TOTAL DERIV.

$$\frac{\delta S}{\delta A} = -d*F + *j = 0$$

$$\boxed{d*F = *j} \qquad \boxed{\partial_{\mu} F^{\mu\nu} = j^{\nu}}$$

OBSERVE : ELECTROMAGNETIC DUALITY
IS HODGE DUALITY

$$*F \sim \varepsilon \dots F^{\cdot\cdot} = \tilde{F}_{..}$$

μ

if this is, eg 01 (E_x)

then this is 23 (B_x)

so: $* : \underline{E} \longleftrightarrow \underline{B}$

MAXWELL IN VACUUM: $dF = 0$, $d*F = 0$ } inv under
$F \leftrightarrow *F$
$E \leftrightarrow B$

one come from geom
other from action principle
... DIDN'T MATTER WHICH, AS LONG AS $j = 0$

IN PRESENCE OF MATTER: $dF = 0$
$d*F = *j$

no longer symmetric. putting j_M in "breaks" geometry ... single potential description fails one way to understand this:

$dF = 0$ came from $d^2 = 0$

$\updownarrow$

in the homology picture, this had to do w/ the <u>boundary</u> ($\partial^2 = 0$)

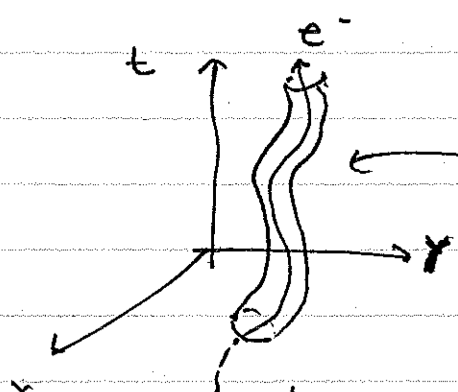
... hm?

can try to diagnose by doing EM transform

<u>BUT</u>: $F \leftrightarrow *F$ ... what happens to A?

only an issue for $j \cdot A$ term

only in "the support" of j



ant tunnel through spacetime around e^- worldline

... outside of this, EM duality is good.

we introduce a <u>topology</u>!

SO, QUALITATIVELY: can try to study mag. monopole by "dualizing" thy w/ only electric charges

→ seems to introduce a topology (due to worldlines that break EM duality)

→ MAG MONOPOLE "LIVES" IN THIS TOPOLOGY

→ WINDING MODE OF GAUGE FIELD

SLIGHTLY MORE QUANTITATIVE: can push through dualization w/ LAGRANGE MULTIPLIER THEN INTEGRATING OUT ORIGINAL GAUGE FIELD DESCRIPTION.

<u>EXAMPLE</u> EM in 2+1 dim

still have $dF = 0$

$$d{*F} = {*j}$$

BUT NOW: $*F$ is A 1-FORM

"COMPLETES" 2-FORM F

$*j$ is a 2-form

$$F = \begin{pmatrix} 0 & E_x & E_y \\ & 0 & B \\ \# & & 0 \end{pmatrix}$$

LET $\boxed{*F = \tilde{F}}$ dual field strength

nb 1 form in 3D still has 3 deg of freedom.

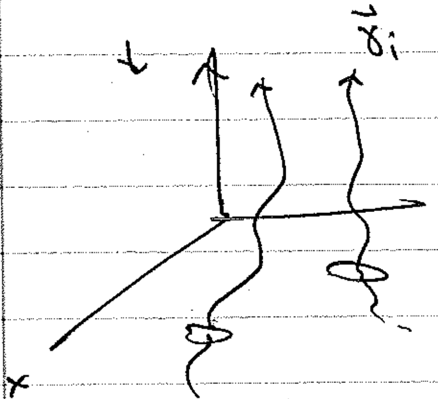
$$d\tilde{F} = d{*F} = {*j} \neq 0$$

so $\tilde{F} \neq d\tilde{A}$

$\tilde{A}$ is A SCALAR POT.

$\tilde{F}_\mu = \partial_\mu \tilde{A}$

the obstruction is the electric current which only exist on the electron worldlines.



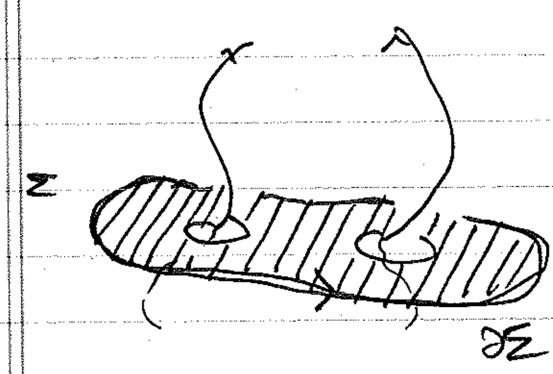
$\star j = \sum_i q_i \, \delta^{(2)}(\vec{\gamma}_i - \vec{x}) \, dx \wedge dy$

$\star j$ is technically a space of measure zero

AWAY FROM $\star j$ (MOST of space) →

$\tilde{F} = d\tilde{A}$

CONSIDER SOME SURFACE PIERCED BY ELECTRON WORLDLINES



$\int_\Sigma d\tilde{F} = \int_\Sigma \star j$

$= \sum_i q_i \int_\Sigma \delta(\gamma - x) \, d^2x$

$= e$ (INTEGER)

BUT : $\int_\Sigma d\tilde{F} = \int_{\partial\Sigma} \tilde{F} = \int_{\partial\Sigma} d\tilde{A}$

BOUNDARY   POTENTIAL

$$= \underbrace{\tilde{A}(\theta = 2\pi) - \tilde{A}(\theta = 0)}$$

should be ZERO,
BUT THIS IS SOME (generally $\neq 0$)
INTEGER!

SO: $\tilde{A}$ IS A PERIODIC POTENTIAL.

"moduli space is a cylinder"

MONOPOLES ARE IDENTIFIED W/ THIS WINDING.