GRADES ARE IN

LEC 17: DIFFERENTIAL FORMS 12 MARCH

LAST 2 LECTURES: a bit more geometry.
"for culture" - since we've built up so much machinery

→ shows up in geometric mechanics ← PHASE SP. momenta = 1-forms
statistical mechanics
→ thermody. potentials are 1-forms
← from S(form)
anything w/ "topology" ← esp GAUGE THY
eg INSTANTONS, ANOMALIES
← chern-simons forms

for simplicity - let's stick to flat space

UPPER vs. LOWER INDICES, redux

Vectors, V^μ — (one) FORMS, ω_μ

tangent space citizen $\in T_pM$ ← EATS VECTOR, SPITS OUT #
- LINEAR MAP FROM $T_pM \to \mathbb{R}$
$\omega(V) = \omega_\mu V^\mu$

SIMILARLY, CAN THINK OF $\omega_\mu V^\mu = V(\omega)$ BY LINEARITY ← so forms & vectors are kind of "the same" right?!?

INDEED, ONCE YOU HAVE A METRIC / inner product, there is a clear duality between upper & lower index objects. THEY CARRY THE SAME DATA.

but, we can add more structure (w/ foresight)

1. K-forms : antisymmetric lower indexed tensors
   eg $A_{\mu_1 \ldots \mu_k}$

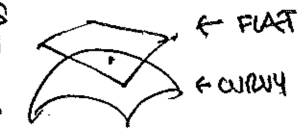
   why? well - this seems important for things like areas & volumes
   eg (AREA) = $\underline{V} \times \underline{W}$ for parallelogram

2. DIFFERENTIAL K-FORMS
   these take special meaning when we go to higher dimensions & want to "do calculus"

   VECTORS ⟷ DIRECTIONS for DIRECTIONAL DERIV.
   (diff) forms ⟷ eat vectors / can be integrated

   d : differential operator

EQUIVAL. PRINCIPLE, MATHEMATICALLY



(LIVING ON THE (CO)TANGENT SPACE OF A POINT ON A MANIFOLD

CAHILL CH. 12

<u>examples of forms</u>

$\omega_{11} = \omega_{12} = 0$ by antisym.

<u>2-form</u>: $\omega(\underbrace{A, B}_{\text{VECTORS IN } \mathbb{R}^2}) = \omega_{11} A^1 B^1 + \omega_{12} A^1 B^2 + \omega_{21} A^2 B^1 + \omega_{22} A^2 B^2$

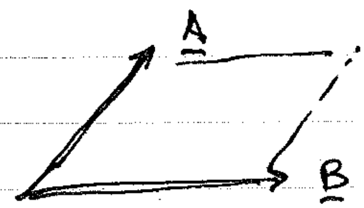
$\omega_{21} = -\omega_{12}$
B/C k-form is totally antisymmetric

$$\omega(A,B) = \omega_{12}\underbrace{(A^1 B^2 - A^2 B^1)}_{\text{DETERMINANT!}}$$

$$= \omega_{12} \begin{vmatrix} A_1 & B^1 \\ A^2 & B^2 \end{vmatrix}$$

↑

<u>oriented</u> area of parallelogram



CAN CHOOSE $\omega_{12} = 1$ s.t. this <u>IS</u> the det.

INDEED, A 3x3 DETERMINANT IS ANOTHER EXAMPLE OF A ~~3~~ FORM, THIS TIME $K=3$.

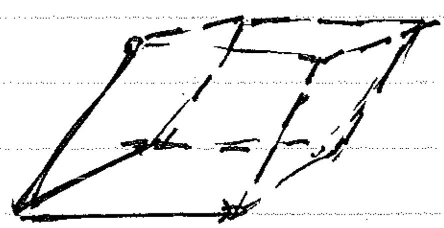
$$\begin{vmatrix} A^1 & B^1 & C^1 \\ A^2 & B^2 & C^2 \\ A^3 & B^3 & C^3 \end{vmatrix} = \omega(A,B,C)$$

for $\omega_{12}=\omega_{23}=\omega_{13}=1$

$$\parallel$$

$$\underline{A}\cdot\underline{B}\times\underline{C}$$

which gives volume of parallelpiped



X PRODUCT IS A SPECIAL OPERATION IN $\mathbb{R}^3$ ... no analog in $\mathbb{R}^2$ or $\mathbb{R}^4$...

easy to see how ω generalizes in higher dimensions.

TO BE EXPLICIT : a k-form / "exterior k-form" is a linear, antisym function of k vectors.

$$\omega(A_1, A_2, \dots, A_k) = -\omega(A_2, A_1, A_3, \dots)$$

((etc))

$$\omega(\alpha A_1 + \beta B_1, A_2, \dots) = \alpha\omega(A_1, \dots) + \beta\omega(B_1, \dots)$$

MAKING BIGGER FORMS: EXTERIOR PRODUCT

$$\wedge : (\text{k-form}), (\text{l-form}) \longrightarrow (\text{k+l})\text{-form}$$

antisym over k indices

antisym over l indices

combined in such a way that this is antisym over (k+l) indices

eg: wedge of 2 1-forms

$$\omega_1 \wedge \omega_2 (A, B) = \underbrace{\omega_1(A)\,\omega_2(B) - \omega_1(B)\,\omega_2(A)}_{\text{antisym by construction}}$$

PRODUCT OF K 1-forms:

$$\omega_1 \wedge \cdots \wedge \omega_k (A_1, \cdots, A_k) = \begin{vmatrix} \omega_1(A_1) & \cdots & \omega_k(A_1) \\ \vdots & & \vdots \\ \omega_1(A_k) & \cdots & \omega_k(A_k) \end{vmatrix}$$

MOST GENERAL k-form takes this structure

$$\omega^{(k)} = \sum_{\substack{i_1, \cdots, i_k \\ k \leq n}} a_{i_1 \cdots i_k} \underbrace{e_{i_1} \wedge \cdots e_{i_k}}_{\text{BASIS 1-FORMS}}$$

$k > n \Rightarrow$ zero by pigeon-hole principle

<u>implication</u>: $\wedge$ is also antisymmetric

$$\omega^{(k)} \wedge \rho^{(l)} = (-)^{kl} \rho^{(l)} \wedge \omega^{(k)}$$

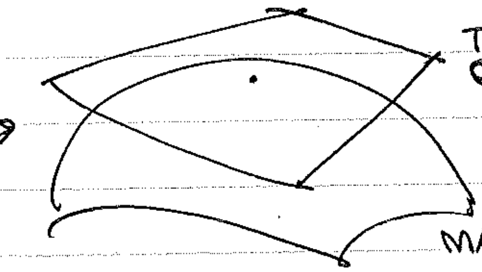
<u>NEXT</u>: how to connect this structure to the "tangent space" (GENERALIZED) of a manifold

BY THE WAY: the decomposition of tensors into antisym. pieces is nothing too obscure...
we know that:

$$\text{Tensor} = \text{Sym} \oplus \text{Antisym} \oplus \text{Trace}$$

TRACELESS

CALCULUS



vectors & stuff live here

↑ base space

...integrate over paths & areas of this

X ~~a useful crutch: imagine M is an n-dim embedding in $\mathbb{R}^{n+1}$~~

~~eg~~ $\cancel{\sum_{i=1}^{n+1} x_i^2 = 1}$

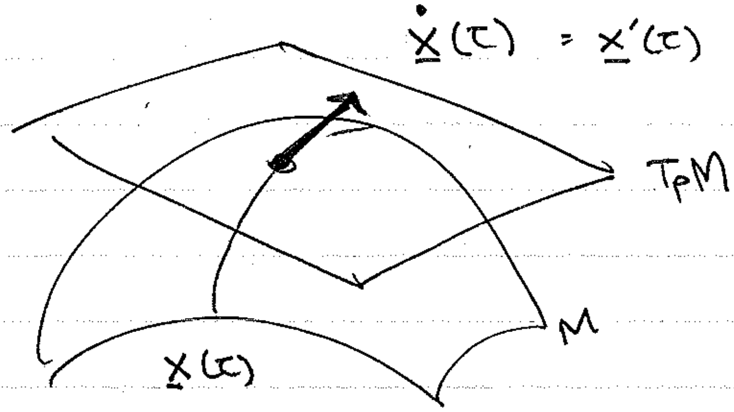
NOW IMAGINE AN (INFINITELY DIFFERENTIABLE) TEST FUNCTION f THAT LIVES ON M

$$f: M \to \mathbb{R}$$

DEFINE – or reacquaint yourself w/ – THE <u>DIFFERENTIAL OPERATOR</u>, acting on the function.

$$df: \text{vector} \to \#$$ ← df is a 1-form

'PHYSICALLY': df takes in a direction (w/ length) (ie velocity vec) & spits out the directional derivative of f along that dir.



PARAM. TRAJECTORY ; $\underline{x}(0) = p$

f is defined on M

components of velocity ve

$$df(\dot{\underline{x}}) = \underbrace{\sum_i \frac{\partial f(x(\tau))}{\partial x^i} \frac{dx^i(\tau)}{d\tau}\Big|_0}_{\frac{d}{d\tau} f(x(\tau))} = \underbrace{\nabla f}(x(\tau)) \cdot \dot{\underline{x}}(\tau)\Big|_0$$

$\dot{x}$

COMPONENTS OF 1-FORM

lower index stuff

the difference in f between ticks along $x(\tau)$.

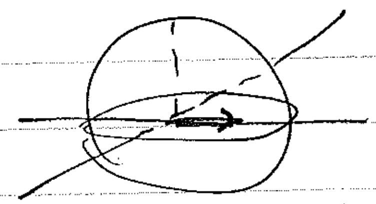
so $df(\dot{\underline{x}})$ is indeed something like a Δf = f(one point) − f(another)

ALSO: RELATION OF "TOTAL DIFFERENTIAL" df & PARTIAL DIFFERENTIAL $(\partial/\partial x^i) f$

THE BASIS 1-FORMS ARE SIMPLY $\underbrace{dx, dy, \ldots}_{dx^i}$

$$df = \sum_i \frac{\partial f(x)}{\partial x^i} dx^i$$

eg: $f(\underline{x}) = x^2 + y^2$



$$df = 2x\, dx + 2y\, dy$$

$$df\left(\left(1,0\right)_p\right) = 2x_{(p)} \underbrace{dx\left(\begin{pmatrix}1\\0\end{pmatrix}\right)}_{1} + 2y_p \cancel{dy\left(\begin{pmatrix}1\\0\end{pmatrix}\right)}^{0}$$

vector

base point

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the abstract way of writing this is

$$df(\partial/\partial x^k) = \partial/\partial x^k f$$

BASIS VEC

s.t. $dx^i(\partial/\partial x^k) = \frac{\partial}{\partial x^k} x^i = \delta^i_k$

BASIS 1-FORM

eg. CHANGE OF VARIABLES: new variables are just functions of old ones...

$$dy^i = \sum_{j=1}^{n} \frac{\partial y^i(x)}{\partial x^j} dx^j$$

← we knew this now that we

UPPER INDEX: $\frac{\partial x^i}{\partial x}$

(btw: BASIS 1-FORM HAS UPPER INDEX; BASIS VEC HAS LOWER)

BASIS K-forms: $\underbrace{dx^{i_1} \wedge \cdots \wedge dx^{i_k}}_{\text{totally antisym.}}$

eg. BASIS 2-forms in $\mathbb{R}^3$:

$dx \wedge dy, \quad dx \wedge dz, \quad dy \wedge dz$

$dx \wedge dx \equiv 0$ by antisym.

so: $\nexists$ k forms on $\mathbb{R}^n$ w/ $n<k$

SO WE HAVE: differentiation op: d : func $\to$ 1-form
form concatenator: $\wedge$: k-form, l-form $\to$ (k+l) form
basis k-forms: $dx^{i_1} \wedge \dots \wedge dx^{i_k}$

generalize differentiation: k-form $\to$ (k+1)-form

func $\equiv$ 0-form.

$$\omega = \sum \omega_{i_1 \cdots i_k}(x) dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

conventionally there are factors of $1/k!$ here b/c of antisymmetry ... for our purposes, it's an overall prefactor that can be absorbed into coefficients

BUT IN ANY REAL CALC — BE CAREFUL!

$$d\omega = \sum \underbrace{\frac{\partial \omega_{i_1 \cdots i_k}(x)}{\partial x^j}}_{\sim\, \omega^{(k+1)}_{i_1 \cdots i_{k+1}}} \underbrace{dx^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}}_{\text{also conventions of ordering here ... just be consistent}}$$

eg. $\mathbb{R}^3$ w/ some 1-form

$$A = A_x\,dx + A_y\,dy + A_z\,dz$$

dA is a 2-form: $\qquad \partial_x A_x\, dx\wedge dx \equiv 0$

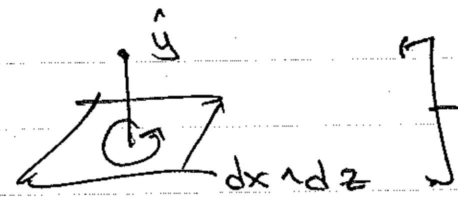
$$\begin{aligned} dA = {} & \partial_y A_x\, dy\wedge dx + \partial_z A_x\, dz\wedge dx \\ & + \partial_x A_y\, dx\wedge dy + \partial_z A_y\, dz\wedge dy \\ & + \partial_x A_z\, dx\wedge dz + \partial_y A_z\, dy\wedge dz \end{aligned}$$

USE ANTISYMM of, eg. $dy\wedge dx = dx\wedge dy$
& GROUP TERMS:

$$\begin{aligned} dA = {} & (\partial_x A_y - \partial_y A_x)\, dx\wedge dy \\ & + (\partial_y A_z - \partial_z A_y)\, dy\wedge dz \\ & + (\partial_z A_x - \partial_x A_z)\, dz\wedge dx \end{aligned}$$

!! it's the curl!

THE CURL ISN'T A VECTOR,
IT'S A 2-FORM WHERE THE
BASIS VECTORS ARE THE
PLANE IN WHICH THERE IS CIRCULATION



what is the relation?

ANOTHER THING WE NOTICE: $d^2 = 0$

eg: $f(x) = x^2 + 2y^2$ in $\mathbb{R}^2$

$df = 2x\,dx + 4y\,dy$

$d^2 f = 2\,dx \wedge dx + 4\,dy \wedge dy = 0$

eg: $f(x) = xy$

$df = x\,dy + y\,dx$

$d^2 f = dx \wedge dy + dy \wedge dx = 0$

<u>indeed</u>: $d\omega = \sum_i \partial_j \omega_{i_1 \cdots i_k}\, dx^j \wedge dx^{i_1} \wedge \cdots dx^{i_j}$

$$d^2\omega = \sum_i \underbrace{\partial_j \partial_\ell}_{\text{sym}} \omega_{i_1 \cdots i_k} \underbrace{dx^\ell \wedge dx^j}_{\text{ANTISYM}} \wedge \cdots$$

$\equiv 0$

So exterior derivative is <u>nilpotent</u>, $d^2 = 0$

so: d of a k-form, ω, is $(k+1)$ form

automatically zero if $(k+1) > n$ $(\mathbb{R}^n)$

$\underline{\omega = d\varphi}$

cohomology

<u>then</u>: if i have a k form ω,
that happens to be $\omega = d\varphi$

φ is kind of a <u>POTENTIAL</u> for ω

if i shift $\underbrace{\varphi \to \varphi}_{(k-1)\text{-form}} + d\underbrace{\alpha}_{(k-2)\text{ form}}$,

$\omega \to \omega$

ie ω is <u>invariant</u>

sounds similar to something, eh?
GAUGE REDUNDANCY.

↳ curl... antisym...

4-VECTOR POTENTIAL

$A = A_\mu dx^\mu$

$F = F_{\mu\nu}\, dx^\mu \wedge dx^\nu = dA$

AUTOMATICALLY ANTISYMMETRIZES

$\underbrace{(\partial_\mu A_\nu)}_{F_{\mu\nu}}\, dx^\mu \wedge dx^\nu$

AGAIN — I'M DROPPING PREFACTORS LIKE $\frac{1}{2!}$ ...

Physics

NILPOTENCY : $F = dA$ ("math.")

then: $A \to \underline{A + d\alpha}$ leaves F invt.

GAUGE TRANSFORMATION

α = 0-form

$d\alpha = \partial_\mu \alpha \ dx^\mu$

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recall curl : relation btw'n

$\underset{\text{vector}}{\underline{\nabla} \times \underline{V}}$ vs. $\underset{\text{1-form}}{dV}$?. in $\mathbb{R}^3$

same components, different basis

given METRIC, easy to see how 1-form $\Leftrightarrow$ vector.

also clear that 2-form in $\mathbb{R}^3$ in some sense encodes same info!

rw. relate to some basis

DEFINE HODGE STAR $*$, by action on basis k-forms

in 4 dimensions

$$* 1 = \frac{1}{4!} \epsilon_{\mu\nu\rho\sigma} dx^\mu \wedge \cdots \wedge dx^\sigma$$
$$* dx^\mu = \frac{1}{3!} g^{\mu\alpha} \epsilon_{\alpha\nu\rho\sigma} dx^\nu \wedge dx^\rho \wedge dx^\sigma$$

$$* dx^\mu \wedge dx^\nu = \frac{1}{2!} g^{\mu\alpha} g^{\nu\beta} \epsilon_{\alpha\beta\rho\sigma} dx^\rho \wedge dx^\sigma$$

& so forth. (I'M PLAYING IT SUPER LOOSE W/ FACTORIALS!)

MORE GENERALLY

$*$ (k-form) = (n-k) form
w/ basis that is complementary to orig k-form.

thus in $\mathbb{R}^3$: $* dx \wedge dy \sim dz$

s.t. $\nabla \times \underline{V} \sim * dV$

__note__: $dx \wedge dy$ is EVEN UNDER PARITY
s.t. dV ———— " ————
WHICH IS DIFFERENT FROM A __VECTOR__
↪ "AXIAL VECTOR"

$*$ is a trick to fill in the rest of the "form-ness"

to "complete" the form to a maximal form: $\Omega = dx^1 \wedge \ldots \wedge dx^n$

"VOLUME FORM"

PREVIEW OF NEXT TIME

$\int \Omega$ = VOLUME

$$* dx^\mu \wedge dx^\nu = \underbrace{\varepsilon_{\alpha\beta\rho\sigma} \, dx^\rho \wedge dx^\sigma \, g^{\alpha\mu} g^{\beta\nu}}_{\varepsilon^{\mu\nu}{}_{\rho\sigma}}$$

$\underline{\text{then}}$: $$*F = \underbrace{\varepsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}}_{} dx^\mu \wedge dx^\nu$$

$\underline{E} \longleftrightarrow \underline{B}$

ELECTROMAGNETIC DUALITY

VOLUME FORM → BORN TO INTEGRATE

$F \wedge {*F}$ is A 4-form on 4-space

$\uparrow$

LAGR. DENSITY

so: $S = \int F \wedge {*F}$

$$= \int \underbrace{\varepsilon_{\mu\nu\rho\sigma}}_{\text{ANTISYM}} F^{\mu\nu} F_{\alpha\beta} \; \underbrace{dx^\rho \wedge dx^\sigma \wedge dx^\alpha \wedge dx^\beta}_{\text{ANTISYM}}$$

eg: $\mu, \nu = 0, 1$

$\varepsilon_{\mu\nu\rho\sigma} \Rightarrow \rho, \sigma = (2,3)$

$\Rightarrow \alpha, \beta = (0,1)$

$$\sim \int F_{\mu\nu} F^{\mu\nu} \, d^4x$$

indeed : EM ACTION ✓

$E^2 + B^2$

<u>WHY NOT</u> $S = \int F \wedge F$?

- dependence on metric?
- total derivative → topological

(BUT U(1) has no topo.)

[next time:
Maxwell's eqns]

$$F \wedge F = F_{\alpha\beta} F_{\rho\sigma} \, dx^\alpha \wedge dx^\beta \wedge dx^\rho \wedge dx^\sigma$$

what does this look like?

$\alpha\beta = 0\ 1$

$dx \wedge \ldots \Rightarrow \rho\sigma = 2,3$

so this is $\underline{E} \cdot \underline{B}$

or, in tensor notation $F_{\alpha\beta} F_{\delta\gamma} \varepsilon^{\alpha\beta\delta\gamma}$

$$\partial_{[\alpha} A_{\beta]} \partial_{[\delta} A_{\gamma]} \varepsilon^{\cdots}$$

eg $\partial_\alpha A_\beta \, \partial_\delta A_\gamma$

$$\partial_\alpha (A_\beta \partial_\delta A_\gamma) - A_\beta \underbrace{\partial_\alpha \partial_\delta}_{\text{sym}} A_\gamma$$

asym

$= 0$.

this term is a total
derivative.

PREP for next lecture

1. REVIEW via: MAXWELL EQ IN FORMS
2. (POINCARÉ LEMMA)
3. GENERALIZED STOKES' THM
4. MAGNETIC MONOPOLES
5. VORTICES IN 3D

MAXWELL $\delta S_{EM} \rightarrow \overset{3}{d}\overset{2}{*}\overset{2}{F} = \overset{3}{*}\overset{1}{j}$

$$\partial_\mu F^{\mu\nu} = j^\nu$$

from variational principle

2 maxwell eqs.

other 2: $dF = 0$

$$F = dA \Rightarrow \boxed{dF \equiv 0}$$

came for free from geometry

Stokes' theorem

something like $\int_V \nabla \cdot E \, d^3\underline{x} = \int_{\partial V} E \cdot dA$

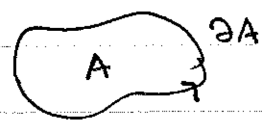
$\uparrow$ INT. OVER BOUNDARY OF V

this is like "int by pts" where derivative moved to volume s.t. $d(\text{vol}) \equiv \partial V$ is the BOUNDARY of V

Green's theorem

$$\int_A (\underline{\nabla} \times \underline{A}) \cdot d\underline{A} = \int_{\partial A} \underline{A} \cdot d\underline{\ell}$$

LINE INTEGRAL



Fundamental thm of calculus

$$\int_I f \, dx = f_{top} - f_{bot} \text{ "} \equiv \int_{\partial I} f \text{ "}$$

Generalized (is simpler)

$$\int_V df \; \underbrace{dx^1 \wedge \dots \wedge dx^{n-1}}_{n-1 \text{ form}}$$

1-form

n- volume

0-form

thm: $= \int_{\partial V} f \; \underbrace{dx^1 \wedge \dots \wedge dx^{n-1}}$

(n-1) volume

n-1 form

& we have seen, eg. that for GREEN'S THM:

$$\int_{area} dA \wedge d\tilde{x}$$

normal 1-form

this gives $(\nabla \times A) \cdot \underbrace{d\tilde{y} \wedge d\tilde{z}}$

AREA

vs: $\int_{\partial(AREA)} A \cdot dl$