LEC 16 : GRAV. WAVES II — 9 MARCH

REVIEW : in the limit $g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)$
$|h_{\mu\nu}| \ll 1$

$$\underbrace{R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R}_{} = -\tfrac{1}{2}\, \partial^2 \bar{h}_{\mu\nu} + \cdots + \mathcal{O}(\bar{h}^2)$$

$G_{\mu\nu}$, LHS OF EINSTEIN EQ

$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu} h$

$\partial^2 = \eta^{\mu\nu}\partial_\mu\partial_\nu$

LINEAR IN h, BUT GOES LIKE DIVERGENCE OF h: $\partial_0 \partial_\alpha \bar{h}^{0\alpha}$

these vanish if

$$\partial_\alpha \bar{h}^{\mu\alpha} = 0$$

want this

then: $-\tfrac{1}{2}\partial^2 \bar{h}_{\mu\nu} = 8\pi G\, T_{\mu\nu}$

wave eqn when $T_{\mu\nu} = 0$

GAUGE FREEDOM

$$x^\mu \to x^\mu + \xi^\mu$$

choose this s.t.

$$\begin{cases} g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \\ |h_{\mu\nu}| \ll 1 \quad \text{is preserved} \end{cases}$$

AND: $\partial_\alpha \bar{h}^{\mu\alpha} = 0$

(4 variables, 4 constraints)

$|\partial_\mu \xi_\nu| \ll 1$

$$\star \quad \bar{h}_{\mu\nu} \to \underbrace{h_{\mu\nu} - \partial_{(\mu}\xi_{\nu)} - \tfrac{1}{2}\eta_{\mu\nu}(h - 2\partial\cdot\xi)}$$

$$\bar{h}_{\mu\nu} - \partial_{(\mu}\xi_{\nu)} + \eta_{\mu\nu}\,\partial\cdot\xi$$

$$\partial_\alpha \bar{h}^{\alpha\mu} \to \partial_\alpha \bar{h}^{\alpha\mu}_{(old)} - \partial_\alpha\partial^\alpha\xi^\mu - \cancel{\partial^\mu \partial\cdot\xi} + \cancel{\partial^\mu \partial\cdot\xi}$$

$$\boxed{\partial^2 \xi^\mu = \partial_\alpha \bar{h}^{\alpha\mu}_{(old)}}$$

SOLUTION EXISTS.

<u>BUT</u>: ACTUALLY: THIS STILL DOES NOT FULLY SPECIFY THE GAUGE (coordinates).

COULD DO AN ADDITIONAL TRANSFORM

$$x^\mu \to x^\mu + \xi^\mu \to (x^\mu + \xi^\mu) + \tilde{\xi}^\mu$$

ALSO w $|\partial_\mu \tilde{\xi}_\nu| \ll 1$
stay in small $h_{..}$ limit

GIVES A SHIFT OF $\partial^2 \tilde{\xi}^\mu$ TO $\partial_\alpha \bar{h}^{\alpha\mu}$

want $= 0$

from following steps on pvs page

So: AS LONG AS $\partial^2 \tilde{\xi}^\mu = 0$, PRESERVE $\partial_\alpha \bar{h}^{\alpha\mu} = 0$

So: in vacuum ($T_{..} = 0$)

POLARIZATION

$$\partial^2 \bar{h}_{\mu\nu} = 0 \Rightarrow \bar{h}_{\mu\nu} = A_{\mu\nu} e^{ik\cdot x}$$

$k^2 = 0$

$$\partial^2 \tilde{\xi}^\mu = 0 \Rightarrow \tilde{\xi}^\mu = B^\mu e^{ik\cdot x}$$

freedom

ROTATE st $K = (\omega, 0, 0, \omega)$

$\bar{h}_{\mu\nu}$ is symmetric $\longrightarrow$ $\boxed{10 \text{ dof}}$ in $A_{\mu\nu}$

$$\partial_\alpha \bar{h}^{\alpha\mu} = 0 \longrightarrow \underbrace{i k_\alpha A^{\alpha\mu}} e^{ik\cdot x} = 0$$

transverse : $\boxed{k_\alpha A^{\alpha\mu} = 0}$ 4 constr.

no comp along last col.

$\boxed{6 \text{ dof}}$ in $A_{\mu\nu}$

from transformation law of $\bar{h}_{\mu\nu}$
(* on page 2)

$$\xi : A_{\mu\nu} \to A_{\mu\nu} - i B_{(\mu} k_{\nu)} + i \eta_{\mu\nu} B\cdot k$$

(I) so: TRACE is : $A^\mu{}_\mu - 2i(B\cdot k) + 4i\,(B\cdot k)$

set equal to zero
(fix, eg. B^0)

(II) $$A_{i0} \to A_{i0} - i B_i \underset{\text{fixed}}{k_0} - i B_0 k_i$$

3 freedoms : set this to 0

$\boxed{2 \text{ dof in } A_{\mu\nu}}$

So: set $\boxed{A_{i0} = 0}$

BUT $K_\alpha A^{\mu\alpha} = 0 \Rightarrow K_\alpha A^{0\alpha} = 0$

then: $K_0 A^{00} + \underbrace{K_i A^{0i}}_{=0} = 0$

$$\Rightarrow \boxed{\partial_t A^{00} = 0}$$

00 comp is TIME-INDEP.

pick $A^{00} = 0$.

then

$$\bar{h}_{\mu\nu} = \begin{pmatrix} 0 & & & \\ & h_+ & h_\times & \\ & h_\times & -h_+ & \\ & & & 0 \end{pmatrix} e^{i\omega(t-z)}$$

(moving in $\hat{z}$ direction)

TRANSVERSE, TRACELESS GAUGE (TT)

SO LET'S APPLY ALL THIS

1. SYMMETRIC:

$$A_{\mu\nu} = \overbrace{\begin{pmatrix} \circ & \circ & \circ & \circ \\ & \circ & \circ & \circ \\ & & \circ & \circ \\ & & & \circ \end{pmatrix}}^{10 \text{ DOF}}$$

by symmetry

2. $K^\alpha A_{\alpha\nu} = K^\alpha A_{\mu\alpha} = 0$

PICK: $K^\alpha = (\omega, 0, 0, \omega)$ $\quad \omega \neq 0$

RECALL: $A_{\mu\nu}$

ROW COLUMN

$K^\alpha A_{\alpha\mu} = 0$ $\qquad\qquad$ $K^\alpha A_{\mu\alpha} = 0$

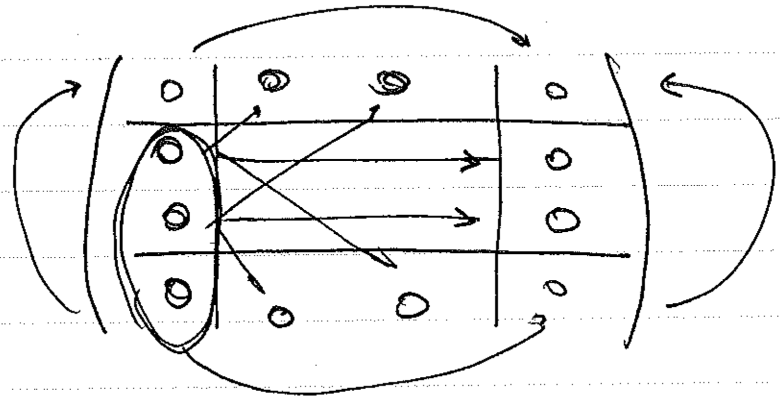
$$\begin{array}{c} \mu=0 \\ \left(\begin{array}{c|cc|c} a & b & c & d \\ \hline & & & \\ \hline -a & -b & -c & -d \end{array}\right) \end{array} \qquad \begin{array}{c} \mu=0 \\ \\ \\ \\ \end{array}\left(\begin{array}{c|cc|c} a & & & -a \\ \hline b & & & -b \\ c & & & -c \\ \hline d & & & -d \end{array}\right)$$

BUT ALSO SYMMETRY:

$$A_{\mu\nu} = \left(\begin{array}{c|cc|c} A & B & C & -A \\ \hline +B & & & -B \\ +C & & & -C \\ \hline -A & -B & -C & A \end{array}\right)$$

$\boxed{6 \text{ DOF}}$

3. $A_{i0} = 0$ (3 EQS)



$\boxed{3 \text{ DOF}}$

4. $A^{\mu}{}_{\mu} = 0$

$\boxed{2 \text{ DOF}}$

$$\begin{pmatrix} 0 & & & \\ & h_+ & h_\times & \\ & h_\times & -h_+ & \\ & & & 0 \end{pmatrix}$$

TRANSVERSE, TRACELESS GAUGE

$$\begin{pmatrix} 0 & & & \\ & h_+ & 0 & \\ & 0 & -h_+ & \\ & & & 0 \end{pmatrix}$$

← stretch x

← shrink y

then $e^{i\omega(t-z)}$ oscillation

$$\begin{pmatrix} 0 & \begin{array}{|cc|} \hline & h_\times \\ h_\times & \\ \hline \end{array} & \\ & & 0 \end{pmatrix}$$ eigenvec: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\uparrow \quad \uparrow$

$+ \quad -$

so this polarization does the same, up to 45° rotation

2 POLARIZATIONS : AS EXPECTED FOR MASSLESS PARTICLE.

TEST PARTICLE IN GRAV. WAVE:

<u>GEODESIC</u>: $\ddot{x}^\mu + \Gamma^\mu_{\rho\sigma} \dot{x}^\rho \dot{x}^\sigma = 0$

$\uparrow$

$\dot{} = d/d\tau$

TEST PARTICLE INIT @ REST: only $\begin{cases} \dot{x}^0 \neq 0 \\ \dot{x}^i = 0 \end{cases}$

$$\ddot{x}^\mu \big|_{t=0} = - \Gamma^\mu_{00} \dot{x}^0 \dot{x}^0$$

$\uparrow$

(really care about $\mu = i$)

$$\Gamma^{\mu}_{00} = \frac{1}{2}\eta^{\mu\nu}\underbrace{(\partial_0 h_{0\nu} + \partial_0 h_{\nu 0} - \partial_\nu h_{00})}_{\equiv 0 \text{ in TT GAUGE}}$$

$\Rightarrow \ddot{x}^{\mu} = 0$

COORDINATE POSITION OF A PIECE OF DUST DOES NOT CHANGE! IT'S STUCK ON THE COORDINATE GRID & IS COMOVING W/ THE GRAV. WAVE.

NOT SURPRISING - LOCAL INERTIAL FRAME (also: $x^0 = \tau$ in free falling coords)

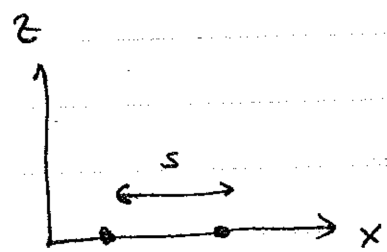
QUICK DIAGNOSIS:

PROPER DIST. VS. COORD DIST.

cf SCHUTZ 9.24

$$\begin{aligned} ds^2 = dt^2 &- (1 - h_+ e^{i\omega(t-z)})\, dx^2 \\ &- (1 + \text{——''——})\, dy^2 \\ &+ 2h_\times e^{i\omega(t-z)}\, dx\, dy \\ &- dz^2 \end{aligned}$$

$\cos\omega(t-z)$



TAYLOR EXP

$s \approx s_0(1 - \frac{1}{2}h_+ \cos\omega t)$

$\Delta s \sim s_0$

BIGGER INIT DISPL → BIGGER

(LIGO should be big)

A GOOD QUESTION: SPACETIME SQUISHES

SO RULERS ALSO SQUISH

SO HOW DO WE MEASURE ANYTHING?

related question: universe expanding
are atoms getting bigger?
(EVEN MORE PRONOUNCED... WOULD IMPLY
CHANGE IN α?!)

SCHUTZ 9.1

ANSWER: TIDAL FORCES / Geodesic deviation

$$\delta \ddot{x}^\mu = R^\mu{}_{\alpha\beta\nu}\, \dot{x}^\alpha \dot{x}^\beta\, \delta x^\nu$$

$\circ = d/d\tau$

$\mathcal{O}(h)$

RECALL: δx^μ IS SEPARATION OF 2 FREE FALLING TEST PARTICLES

LOCALLY INERT FRAME: $\dot{x}^\alpha = (1,0,0,0) + \mathcal{O}(h)$

$$d/d\tau = \partial/\partial t$$

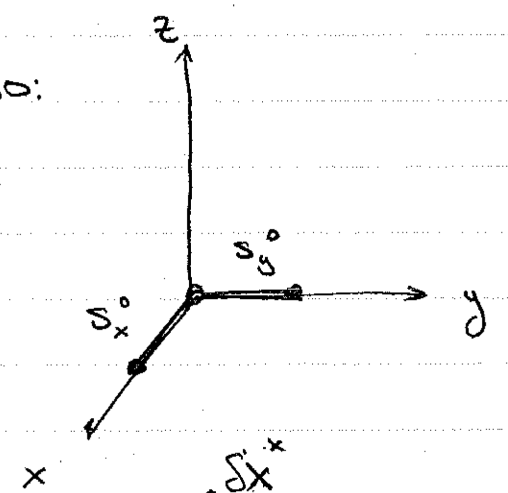
$$\frac{\partial^2 \delta x^\mu}{\partial t^2} = R^\mu{}_{tt\nu}\, \delta x^\nu = -R^\mu{}_{t\nu t}\, \delta x^\nu$$

for TT gauge (ALWAYS ASSUMED FOR US)
& wave in $\hat{z}$ direction

$$R^x{}_{txt} = -\tfrac{1}{2}\partial_t^2 h_+$$
$$R^y{}_{txt} = -\tfrac{1}{2}\partial_t^2 h_\times$$
$$R^y{}_{tyt} = +\tfrac{1}{2}\partial_t h_+$$

recall: $\Gamma \sim \partial g$
$R^{\cdot}{}_{\dots} \sim \partial\Gamma + \cancel{\Gamma^2}_{O(h^2)}$

so:



δx^x

$$\partial_t^2 \delta_x = \tfrac{1}{2} S_x^0\, \partial_t^2 h_+$$
$$\partial_t^2 S_y = -\tfrac{1}{2} S_y^0\, \partial_t^2 h_+$$

← consistent w/ metric analysis (A) on p.9

<u>BUT</u>: this is not a solution, it's a diff eq. encoding GRAV (tidal) force

including ELECTROMAG FORCE:

usually this wins

$$\partial_t^2 \delta x^i = -R^i{}_{0j0}\,\delta x^j + \frac{1}{m_B} F^i_B$$

SO: SINCE ELECTROMAG / NUCLEAR FORCES ARE STRONGER THAN GRAVITY (typically) THE POTENTIAL THAT ARRANGES LATTICE OF p^+, e^-, etc INTO A PHYSICAL "RULER" DOMINATE OVER GRAV. TIDAL FORCES.

in other words: BOHR RADIUS DOESN'T EXPAND W/ SPACETIME.

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from source to observer: GREEN'S FUNCTION

$$\bar{h}_{\mu\nu}(x) = -16\pi G_N \int G(x-y)\, T_{\mu\nu}(y)\, d^4x$$

GREEN'S FUNCTION
$\partial^2 G(x-y) = \delta^{(4)}(x-y)$

SOURCE

CHECK: ∂^2 OF THIS EQ. GIVES EINSTEIN EQ. IN LORENZ GAUGE.

$$G(x-y) = \frac{1}{4\pi|\underline{x}-\underline{y}|}\,\delta\left(|\underline{x}-\underline{y}| - (x^0-y^0)\right)\Theta(x^0-y^0)$$

PROPAG. @ SPEED OF LIGHT

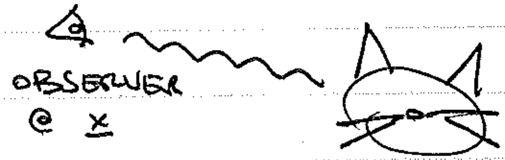
CAUSAL

carroll p.300

$$\bar{h}_{\mu\nu}(t,x) = 4G_N \int \frac{1}{|\underline{x}-\underline{y}|} T_{\mu\nu}(\overbrace{t-|\underline{x}-\underline{y}|}^{t_r \text{ RETARDED TIME}}, \underline{y})\, d^3y$$

sample source only over regions on the light cone.

USUAL eg:



USEFUL FACTS

(i) IN TT GAUGE, ONLY $\bar{h}_{ij} \neq 0$ $(i,j = x,y)$
SO ONLY NEED T_{ij}

(ii) $\partial_\alpha \bar{h}^{\alpha\mu} = 0 \Rightarrow \partial_t \bar{h}^{t\mu} + \partial_i \bar{h}^{i\mu} = 0$

$\Rightarrow \partial_i \bar{h}^{i\mu} = -i\omega \bar{h}^{t\mu}$

(iii) $\underbrace{\int \partial_k (y^i T^{kj})\, d^3y}_{=0}$, $\partial_k = \frac{\partial}{\partial y^k}$

$$\int \partial_k (y^i T^{kj}) = \underbrace{\int T^{ij} d^3y + \int y^i \partial_k T^{kj}\, d^3y}$$

$$\int T^{ij} d^3y = -\int y^i \partial_k T^{kj}\, d^3y$$

<u>Thus</u>: $\int d^3y\, T^{ij} = -\int d^3y\; y^i \partial_k T^{kj}$ (iii)

$= +i\omega \int d^3y\; y^i\, T^{0j}$ (ii)

$= \frac{i\omega}{2} \int d^3y \left(y^i T^{0j} + y^j T^{0i} \right)$

B/C LHS MANIFESTLY SYMM.

$= \frac{i\omega}{2} \int d^3y \left[\cancel{\partial_k \left(y^i y^j T^{0k} \right)} - y^i y^j \partial_k T^{0k} \right]$ → 0, SURFACE

BY EXPLICIT CALC & COMP. TO PREV LINE (OR AGAIN USING (iii), "REVERSE INT. BY PTS)

(ii) AGAIN $= \frac{\omega^2}{2} \int d^3y\; y^i y^j\, \underbrace{T^{00}}_{\rho}$

sinusoidal, or $d^2/dt^2 \neq 0$

time varying I_{ij} → grav waves

I_{ij}, <u>QUADRUPOLE MOMENT</u>

& actually: we know that only <u>TRACE FREE PART</u> CONTRIBUTES

nb: δ_{ij} is only spher. sym tensor.

$I_{ij} \to \mathcal{I}_{ij} = \int d^3y \left(y^i y^j - \frac{1}{3} y^2 \delta_{ij} \right) T^{00}$

OBSERVE :

EM RADIATION : DIPOLE RADIATION $\nearrow q\underline{x}$

b/c cons. of charge $\rightarrow$ no monopole

$\dot{q} = 0$

GRAVITATIONAL RADIATION : QUADRUPOLE

b/c cons. of mass $\longrightarrow$ no monopole

cons of momentum $\rightarrow$ no dipole

ALSO: tells us what kinds of events ...

eg spherical collapse can be

dramatic; eg neutron star $\rightarrow$ BH

BUT: SPHER. SYM. ACCEL

WOULD GIVE A TRACE

TERM IN $\bar{h}_{\mu\nu}$ —

WE GAUGED IT AWAY

(unphysical for grav. waves)

NICE SYSTEMS: BINARY BLACK HOLES

ENERGY RADIATED → what is $T_{\mu\nu}$ of GRAV WAVE?

$$g_{\mu\nu} = \underbrace{g^{(b)}_{\mu\nu}}_{= \eta_{\mu\nu} + \mathcal{O}(h^2)} + h_{\mu\nu}$$

$$R_{\mu\nu} = \underbrace{R^{(b)}_{\mu\nu}}_{\mathcal{O}(1)\ \text{\&}\ \mathcal{O}(h^2)} + \underbrace{R^{(1)}_{\mu\nu}}_{\mathcal{O}(h)} + \underbrace{R^{(2)}_{\mu\nu}}_{\mathcal{O}(h^2)} + \cdots$$

for $T_{\mu\nu} = 0$, EINSTEIN EQ IS $R_{\mu\nu} = 0$

$\mathcal{O}(h)$: $R^{(1)}_{\cdot\cdot} = 0$

$\mathcal{O}(1) + \mathcal{O}(h^2)$: $R^{(b)}_{\mu\nu} + R^{(2)}_{\mu\nu} = 0$

$$R^{(b)}_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}R^{(b)} = \underbrace{-\left(R^{(2)}_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}R^{(2)}\right)}_{8\pi G\, t_{\mu\nu}}$$

$t_{\mu\nu}$: ENERGY MOMENTUM TENSOR OF GRAV WAVE

$$t_{\mu\nu} = \frac{-1}{8\pi G}\left(\langle R^{(2)}_{\mu\nu}\rangle - \frac{1}{2}\eta_{\mu\nu}\langle R^{(2)}\rangle\right)$$

AVERAGE OVER CYCLE &
FINITE VOLUME

CHENG
§15-4

PLUGGING & CHUGGING

$$t_{00} = \frac{1}{16\pi G}\left\langle (\partial_0 \tilde{h}_+)^2 + (\partial_0 \tilde{h}_\times)^2 \right\rangle$$

$$\tilde{h} = h\cos[\omega(t-z)]$$

<u>REMARK</u>: HULSE-TAYLOR BINARY PULSAR
INDIRECT EVIDENCE FOR GRAV RAD

DECAY OF PERIOD MEASURED
& CONSISTENT W/ ENERGY
RADIATED FROM GRAV. RAD.